CHAPTER I

SPHERICAL TRIGONOMETRY

1. *Introduction.*

When we look at the stars on a clear night we have the familiar impression that they are all sparkling points of light, apparently situated on the surface of a vast sphere of which the individual observer is the centre. The eye, of course, fails to give any indication of the distances of the stars from us; however, it allows us to make some estimate of the angles subtended at the observer by any pairs of stars and, with suitable instruments, these angles can be measured with great precision. Spherical Astronomy is concerned essentially with the *directions* in which the stars are viewed, and it is convenient to define these directions in terms of the positions on the surface of a sphere—the *celestial sphere*—in which the straight lines, joining the observer to the stars, intersect this surface. It is in this sense that the usual expression "the position of a star on the celestial sphere" is to be interpreted. The radius of the sphere is entirely arbitrary. The foundation of Spherical Astronomy is the geometry of the sphere.

2. *The spherical triangle.*

Any plane passing through the centre of a sphere cuts the surface in a circle which is called a *great circle*. Any other plane intersecting the sphere but not passing through the centre will also cut the surface in a circle which, in this case, is called a *small circle*. In Fig. 1, EAB is a great circle, for its plane passes through O, the centre of the sphere. Let QOP be the diameter of the sphere perpendicular to the plane of the great circle EAB. Let R be any point in OP and suppose a plane drawn through R parallel to the plane of EAB; the surface of the sphere is then intersected in the small circle FCD. It follows from the construction that OP is also perpendicular to the plane of FCD. The extremities P and Q of the common perpendicular diameter QOP are called the *poles* of the great circle and of the parallel small circle. Now let $PCAQ$ be any great circle passing through the poles P and Q and intersecting the small circle FCD and the great circle EAB in C and A respectively. Similarly, PDB is part of another great circle passing through P and Q. We shall find it convenient to refer to a particular great circle by specifying simply any portion of its circumference. When two great circles intersect at a point they are said to include a *spherical angle* which is defined as follows. Consider the two great circles PA and PB intersecting at P. Draw PS and PT, the tangents to the

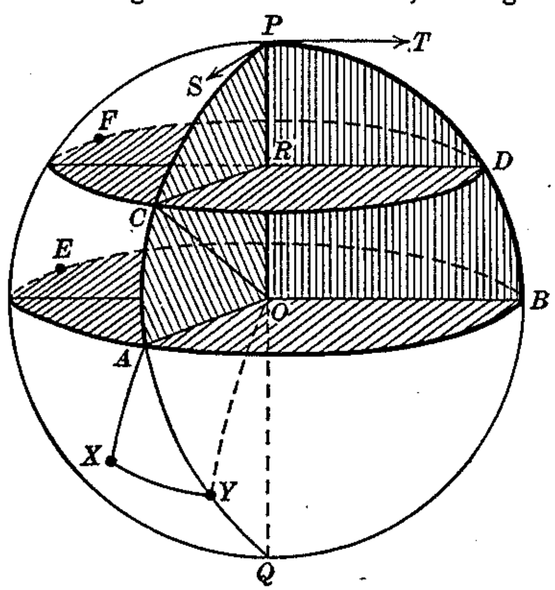

Fig. 1.

circumferences of PA and PB respectively. PT is, by construction, perpendicular to the radius OP of the great circle PB and, being in the plane PBO, is therefore parallel to the radius OB. Similarly PS is parallel to the radius OA. The angle SPT defines the spherical angle at P between the two great circles PA and PB, and it is equal to the angle AOB, AB being the arc intercepted on the great circle, of which P is the pole, between the two great circles PA and PB. It is to be emphasised that a spherical angle is defined only with reference to two intersecting *great circles*.

If we are given any three points on the surface of a sphere, then the sphere can be bisected so that all three points lie in the same hemisphere. If the points are joined by great circle arcs all lying on this hemisphere the figure obtained is called a *spherical triangle*. Thus, in Fig. 1, the three points A, X and Y on the spherical surface are joined by great circle arcs to form the spherical triangle AXY. AX, AY and XY are the *sides* and the spherical angles at A, X and Y are the angles of the spherical triangle. Actually, if R is the radius of the sphere, the length of the spherical arc AY is given by

$$AY = R \times \text{angle } AOY,$$

the angle AOY being expressed in circular measure, i.e. in radians. Now for all great circle arcs on the sphere the radius R is constant and it is convenient to consider its length as unity. The arc AY is then simply the angle which it subtends at the centre of the sphere. If AY is, let us say, one-eighth of the circumference of the complete great circle through A and Y, the side AY is then $\frac{\pi}{4}$ in circular measure and there is no ambiguity if it is expressed as 45°; similarly, for the remaining sides of the triangle. It follows from the definition of a spherical triangle that no side can be equal to or greater than 180°. As another example, PAB is a spherical triangle two of whose sides PA and PB each subtend $\frac{\pi}{2}$ radians or 90° at O; in this instance we say that PA and PB are each equal to $\frac{\pi}{2}$ radians or 90°. But PCD is *not* a spherical triangle, for the arc CD is not a part of a great circle. Accordingly, the formulae which will be derived for spherical triangles will not be applicable to such a figure as PCD.

3. *Length of a small circle arc.*

Consider, in Fig. 1, the small circle arc CD. Its length is given by

$$CD = RC \times \text{angle } CRD.$$

Also, the length of the spherical arc AB is given by

$$AB = OA \times \text{angle } AOB.$$

But since the plane of FCD is parallel to the plane of EAB, then $C\hat{R}D = A\hat{O}B$, for RC, RD are respectively parallel to OA, OB.

Therefore $$CD = \frac{RC}{OA} . AB.$$

But, since $OA = OC$ (radii of the sphere), we have

$$CD = \frac{RC}{OC} . AB.$$

Now RC is perpendicular to OR; $\therefore RC = OC \cos R\hat{C}O$. From the parallelism of RC and OA, $R\hat{C}O = A\hat{O}C$. Hence

$$CD = AB \cos A\hat{O}C.$$

Now AOC is the angle subtended at the centre of the sphere by the great circle arc AC. The formula can then be written as

$$CD = AB \cos AC,$$

or, since $PA = 90°$, $$CD = AB \sin PC \qquad ......(1).$$

4. *Terrestrial latitude and longitude.*

The concepts introduced so far will now be illustrated with reference to the earth. For many practical problems, the earth can be regarded as a spherical body spinning about a diameter PQ (Fig. 2). P is the *north pole* and Q is the *south pole*. The great circle whose plane is perpendicular to PQ is called the *equator*. Any semi-great circle terminated by P and Q is a

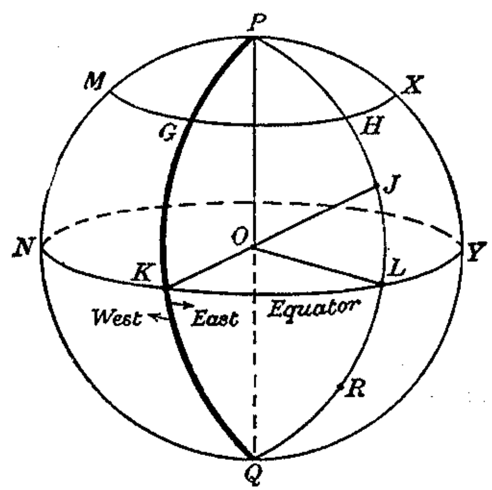

Fig. 2.

meridian. In particular, the meridian which passes through the fundamental instrument (the transit circle) of Greenwich Observatory is, by universal agreement, regarded as the principal or standard meridian; let it be $PGKQ$ in Fig. 2, intersecting the equator in K. Let $PHLQ$ be any other meridian cutting the equator in L. The angle KOL is defined to be the *longitude* of the meridian PHQ and it can be described equally well as the equatorial arc KL or the spherical angle KPL. Longitudes are measured from 0° to 180° *east* of the Greenwich meridian and from 0° to 180° *west*, following the directions of the arrows near K in Fig. 2. Thus, from the figure, the longitude of the meridian PHQ is about 100° east (E) and that of the meridian PMQ is about 60° west (W). All places on the same meridian have the same longitude and the meridian on which a particular place is situated is specified with reference to the principal meridian PGQ. To specify completely the position of a place on the surface of the earth, we require to describe its position on its meridian of longitude. This is done with reference to the equator. Consider a place J on the meridian PHQ. The meridian through J cuts the equator in L and the angle LOJ, or the great circle arc LJ, is called the *latitude* of J. If J is between the equator and the north pole P, as in Fig. 2, the latitude is said to be north (N); a place such as R, between the equator and the south pole Q, is said to be in south latitude (S). In this way the position of any point on the surface of the earth is referred to the two fundamental great circles, the equator and the meridian of Greenwich.

Let ϕ denote the latitude of J; then $L\hat{O}J$ or $LJ = \phi$. Since OP is perpendicular to the plane of the equator, $P\hat{O}L = 90°$ and therefore $POJ = 90° - \phi$. The angle POJ or the spherical arc PJ is the *colatitude* of J. We have thus

$$\text{Colat.} = 90° - \text{Lat.}$$

All places which have the same latitude lie on a small circle parallel to the equator, called a *parallel of latitude*. Thus all places with the same latitude as Greenwich lie on the small circle $MGHX$. If θ denotes the latitude of Greenwich, then by formula (1) the length of the small circle arc HX, for example, is given in terms of the length of the corresponding equatorial arc LY by

$$HX = LY \cos \theta \qquad ......(2).$$

To give greater precision to the meaning of this formula, we consider the units in which distances on the surface of the earth are expressed. The simplest is that defined as the great circle distance between two points subtending an angle of one minute of arc at the centre of the earth—this unit is known as the *nautical mile* and is equivalent to 6080 feet (we neglect the small variations in this value due to the fact that the earth is not quite a sphere). If the difference in longitude between any two places on the same parallel of latitude is known, e.g. LY, then LY can be expressed as so many minutes of arc and this number is the number of nautical miles between the two points L and Y on the equator. The formula (2) then provides the means of calculating the distance between H and X expressed in nautical miles (or minutes of arc) and *measured along the parallel of latitude.*

5. *The fundamental formula of spherical trigonometry.*

Let ABC be a spherical triangle (Fig. 3). Denote the sides BC, CA, AB by a, b and c respectively. Then, by our definition, the

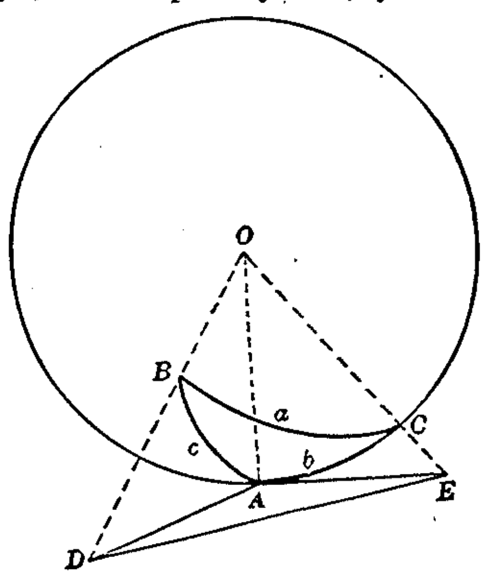

Fig. 3.

side a is measured by the angle BOC subtended at the centre O of the sphere by the great circle arc BC. Similarly, b and c are measured respectively by the angles AOC and AOB. Let AD be the tangent at A to the great circle AB and AE the tangent at A to the great circle AC. Then the radius OA is perpendicular to AD and AE. By construction, AD lies in the plane of the great circle AB; hence, if the radius OB is produced, it will intersect the tangent AD at a point D. Similarly, the radius OC when produced will meet the tangent AE in E. Now the spherical angle BAC is defined to be the angle between the tangents at A to the great circles AB and AC, so that the spherical angle $BAC = D\hat{A}E$. The spherical angle BAC will be denoted simply by A, so that $D\hat{A}E = A$.

Now, in the plane triangle OAD, $O\hat{A}D$ is 90° and $A\hat{O}D$, identical with $A\hat{O}B$, is c. We have then

$$AD = OA \tan c; \quad OD = OA \sec c \qquad \text{......(3).}$$

From the plane triangle OAE we have, similarly,

$$AE = OA \tan b; \quad OE = OA \sec b \qquad \text{......(4).}$$

From the plane triangle DAE we have

$$DE^2 = AD^2 + AE^2 - 2AD.AE \cos D\hat{A}E,$$

or

$$DE^2 = OA^2 [\tan^2 c + \tan^2 b - 2 \tan b \tan c \cos A] \qquad \text{......(5).}$$

From the plane triangle DOE,

$$DE^2 = OD^2 + OE^2 - 2OD.OE \cos D\hat{O}E.$$

But $D\hat{O}E = B\hat{O}C = a$;

$$\therefore \; DE^2 = OA^2 [\sec^2 c + \sec^2 b - 2 \sec b \sec c \cos a] \qquad \text{......(6).}$$

Hence, from (5) and (6),

$$\sec^2 c + \sec^2 b - 2 \sec b \sec c \cos a = \tan^2 c + \tan^2 b - 2 \tan b \tan c \cos A.$$

Now $\quad \sec^2 c = 1 + \tan^2 c; \quad \sec^2 b = 1 + \tan^2 b,$

and after some simplification we obtain

$$\cos \mathbf{a} = \cos \mathbf{b} \cos \mathbf{c} + \sin \mathbf{b} \sin \mathbf{c} \cos \mathbf{A} \qquad \text{......(A).}$$

This is the fundamental formula of spherical trigonometry and it will be referred to in the following pages as the *cosine-formula* or formula **A**. There are clearly two companion formulae; they are

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \qquad \text{......(7)},$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad \text{......(8)}.$$

From the three formulae—**A**, (7) and (8)—all the other formulae of spherical trigonometry in use can be derived. The fundamental formula has two direct practical applications:

(1) *If two sides, e.g.* b *and* c, *and the included angle* A *of a spherical triangle* ABC *are known*, formula **A** enables the calculation of the third side a to be made.

(2) *If all three sides are known*, the angles of the triangle can be found successively by means of **A**, (7) and (8).

For suppose the value of A is required; then by **A**

$$\cos A = \operatorname{cosec} b \operatorname{cosec} c\,[\cos a - \cos b \cos c] \text{ ......(9)}.$$

Formula (9) can be replaced by one more suitable for logarithmic calculations as follows. Since $\cos A = 1 - 2\sin^2 \frac{A}{2}$, we have, from **A**,

$$\cos a = \cos b \cos c + \sin b \sin c \left(1 - 2\sin^2 \frac{A}{2}\right)$$
$$= \cos (b - c) - 2 \sin b \sin c \sin^2 \frac{A}{2},$$

or $$\cos (b - c) - \cos a = 2 \sin b \sin c \sin^2 \frac{A}{2};$$

$$\therefore\ 2 \sin \frac{a + (b - c)}{2} \sin \frac{a - (b - c)}{2} = 2 \sin b \sin c \sin^2 \frac{A}{2}.$$

Let s be defined by $$2s = a + b + c \qquad \text{......(10)}.$$

Then $a + b - c = 2(s - c)$ and $a - b + c = 2(s - b)$.

Hence $$\sin (s - b) \sin (s - c) = \sin b \sin c \sin^2 \frac{A}{2};$$

$$\therefore\ \sin \frac{A}{2} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}} \qquad \text{......(11)}.$$

This form is useful in numerical work. There are two similar equations giving $\sin \frac{B}{2}$ and $\sin \frac{C}{2}$.

If we write $\cos A = 2\cos^2 \frac{A}{2} - 1$ in the formula **A** and proceed as before, we shall obtain

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}} \qquad \text{......(12)}$$

with two similar equations giving $\cos \frac{B}{2}$ and $\cos \frac{C}{2}$.

From (11) and (12) by division we have

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \qquad \text{......(13)}.$$

There are two similar equations, giving $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$.

Any one of (11), (12) and (13) can be used to calculate A, the three sides being known.

6. *The sine-formula.*

We shall now derive what is known as the sine-formula. From the cosine-formula **A**, we have

$$\sin b \sin c \cos A = \cos a - \cos b \cos c.$$

By squaring, we obtain

$$\sin^2 b \sin^2 c \cos^2 A = \cos^2 a - 2 \cos a \cos b \cos c + \cos^2 b \cos^2 c.$$

The left-hand side can be written

$$\sin^2 b \sin^2 c - \sin^2 b \sin^2 c \sin^2 A,$$

or $$1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - \sin^2 b \sin^2 c \sin^2 A.$$

Hence

$$\sin^2 b \sin^2 c \sin^2 A = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

Let a positive quantity X be defined by

$$X^2 \sin^2 a \sin^2 b \sin^2 c = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

Then, from the previous equation,

$$\frac{\sin^2 A}{\sin^2 a} = X^2,$$

so that $$X = \pm \frac{\sin A}{\sin a}.$$

But in a spherical triangle the sides are each less than 180°, and this applies also to the angles. As $\sin \theta$ is positive for all values of θ between 0° and 180°, the minus sign in the above equation is inadmissible, and we have

$$X = \frac{\sin A}{\sin a}.$$

By treating (7) and (8) in a similar way, we shall obtain

$$X = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Hence $$\frac{\sin \mathbf{A}}{\sin \mathbf{a}} = \frac{\sin \mathbf{B}}{\sin \mathbf{b}} = \frac{\sin \mathbf{C}}{\sin \mathbf{c}} \qquad \text{......(B).}$$

This result we shall refer to as the *sine-formula* or formula **B**.

Formula **B** gives a relation between any two sides of a triangle and the two angles *opposite* these sides. It has to be used, however, with circumspection in numerical calculations; for, suppose that the two sides a and b and the angle B are given, then by **B**

$$\sin A = \frac{\sin a \sin B}{\sin b},$$

from which the value of $\sin A$ can be calculated. But $\sin(180° - A) = \sin A$, and without further information it is not possible to decide which of the two angles A or $180° - A$ represents the correct solution. The analogous ambiguity in plane trigonometry may be recalled to the reader's attention.

7. *Formula* **C**.

Write equation (7) in the form

$$\sin c \sin a \cos B = \cos b - \cos c \cos a$$
$$= \cos b - \cos c(\cos b \cos c + \sin b \sin c \cos A)$$
$$= \sin^2 c \cos b - \sin b \sin c \cos c \cos A.$$

Hence, dividing by $\sin c$, we have

$$\sin \mathbf{a} \cos \underline{\mathbf{B}} = \cos \underline{\mathbf{b}} \sin \mathbf{c} - \sin \underline{\mathbf{b}} \cos \mathbf{c} \cos \mathbf{A} \quad \text{...(C),}$$

a relation involving all three sides and two angles.

We can easily prove in a similar manner, beginning with equation (8), that

$$\sin a \cos \underline{C} = \cos \underline{c} \sin b - \sin \underline{c} \cos b \cos A \quad \text{...(14).}$$

If we regard b and c as the two principal sides then A is the contained angle. As we have seen, the cosine-formula **A** gives $\cos a$ in terms of b, c and the included angle A. Formulae **C** and

(14) are, in some ways, analogous to **A** as they give $\sin a \times$ cosine of one of the two angles B and C, adjacent to the side a, in terms of b, c and A.

The formula **C** can also be proved as follows. Suppose the side c of the triangle ABC to be less than 90° (the case when c is between 90° and 180° is left as an exercise to the student). Produce the great circle arc BA to D so that BD is 90° (Fig. 4).

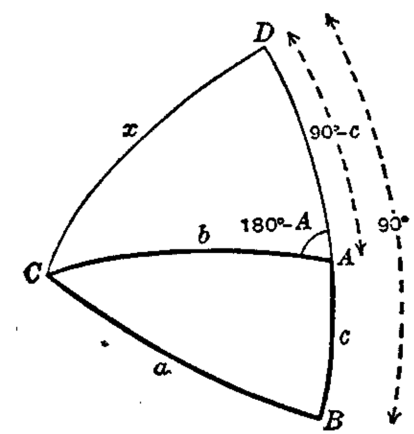

Fig. 4.

Then $AD = 90° - c$ and $C\hat{A}D = 180° - A$. Join C and D by a great circle arc and denote it by x. From the triangle DAC, by **A**,

$$\cos x = \cos(90° - c)\cos b + \sin(90° - c)\sin b \cos(180° - A),$$

or

$$\cos x = \sin c \cos b - \cos c \sin b \cos A \qquad \text{......(15).}$$

From the triangle DBC, by **A**,

$$\cos x = \cos 90° \cos a + \sin 90° \sin a \cos B,$$

or $$\cos x = \sin a \cos B \qquad \text{......(16),}$$

and therefore from (15) and (16)

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A,$$

which is formula **C**.

8. *The four-parts formula.*

Another useful formula, known as the four-parts formula, will now be derived. In the spherical triangle ABC (Fig. 5) consider the four consecutive parts B, a, C, b. The angle C is contained by the two sides a and b and is called the "inner angle". The side a is flanked by the two angles B and C and is called the "inner side". Introduce B and C by means of the cosine-formula; then we have

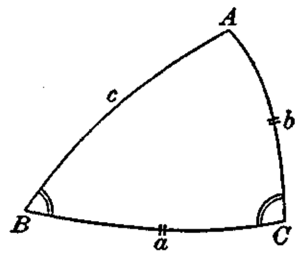

Fig. 5.

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \quad ......(17),$$
$$\cos c = \cos b \cos a + \sin b \sin a \cos C \quad ......(18).$$

Substitute the value of $\cos c$ given by (18) on the right-hand side of (17); then

$$\cos b = \cos a\,(\cos b \cos a + \sin b \sin a \cos C) + \sin a \sin c \cos B;$$
$$\therefore\ \cos b \sin^2 a = \cos a \sin b \sin a \cos C + \sin a \sin c \cos B.$$

Divide throughout by $\sin a \sin b$; then

$$\cot b \sin a = \cos a \cos C + \frac{\sin c}{\sin b} \cos B.$$

But by the sine-formula **B**,

$$\frac{\sin c}{\sin b} = \frac{\sin C}{\sin B}.$$

Hence $$\mathbf{\cos a \cos C = \sin a \cot b - \sin C \cot B} \quad ......(\mathbf{D}),$$

which may be put into words, as an aid to the memory, as follows:

cos (*inner side*).cos (*inner angle*)
= sin (*inner side*).cot (*other side*)
− sin (*inner angle*).cot (*other angle*).

9. *Alternative proofs of the formulae* **A**, **B** *and* **C**.

The formulae **B**, **C** and **D** have been derived by algebraic transformations of the fundamental formula. Another proof of each of **A**, **B** and **C** will now be briefly obtained from a simple and instructive geometrical construction. Let ABC (Fig. 6) be a spherical triangle and O the centre of the sphere. Join O to the vertices and take any point P in OC. From P draw PQ perpendicular to OA and PR perpendicular to OB. In the plane OAB, draw QS perpendicular to OA and RS perpendicular to OB. These perpendiculars meet in S. Join PS and OS. If we draw tangents at A to the great circle arcs AB and AC, these tangents, by definition, include the spherical angle A. But QS and QP are by construction parallel to these tangents. Hence $P\hat{Q}S = A$. Similarly $P\hat{R}S = B$. Also $C\hat{O}B = a$, $C\hat{O}A = b$ and $A\hat{O}B = c$.

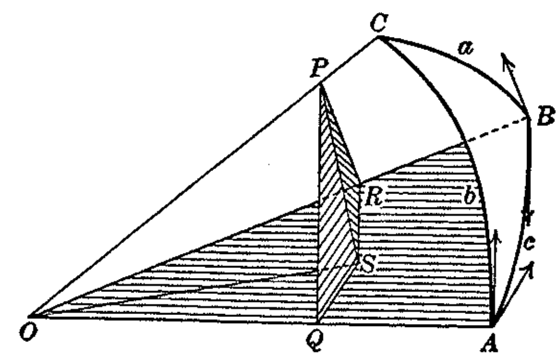

Fig. 6.

The first step is to prove that PS is perpendicular to the plane AOB. By the construction, OQ is perpendicular to both PQ and QS; hence OQ is perpendicular to the plane PQS; therefore OQ is perpendicular to PS which is a line lying in the plane PQS. Similarly, OR is perpendicular to PS. Thus PS is perpendicular to both OQ and OR and is therefore perpendicular to every line in the plane of OQ and OR, that is, PS is perpendicular to the plane OAB and, in particular, to OS, SQ and SR. Thus PQS and PRS are right-angled triangles.

(1) We have, from the right-angled triangles OQP and ORP,

$$PQ = OP \sin b; \quad PR = OP \sin a \quad ......(19).$$
$$OQ = OP \cos b; \quad OR = OP \cos a \quad ......(20).$$

Let x denote the angle SOQ; then $R\hat{O}S = c - x$.

Now $$OS = OQ \sec x \text{ and } OS = OR \sec (c - x).$$

Hence $$OR \cos x = OQ \cos (c - x);$$

$\therefore$ by (20), $$OP \cos a \cos x = OP \cos b \cos (c - x);$$

$$\therefore\ \cos a = \cos b \cos c + \cos b \sin c \tan x.$$

But $$\tan x = \frac{QS}{OQ} = \frac{PQ\cos A}{OQ} = \tan b \cos A,$$

and hence $$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which is formula **A**.

(2) Again, from the right-angled triangles PQS and PRS,

$$PS = PQ \sin PQS = PQ \sin A,$$

and $$PS = PR \sin PRS = PR \sin B.$$

Hence $$PQ \sin A = PR \sin B,$$

and $\therefore$ by (19),

$$OP \sin b \sin A = OP \sin a \sin B,$$

from which formula **B** follows.

(3) We have, from the right-angled triangles OSQ and OSR,

$$QS = OS \sin x \quad \text{and} \quad RS = OS \sin (c - x);$$

$$\therefore \; RS \sin x = QS (\sin c \cos x - \cos c \sin x),$$

or $$RS = QS (\sin c \cot x - \cos c).$$

Now $$RS = PR \cos B = OP \sin a \cos B,$$

and $$QS = PQ \cos A = OP \sin b \cos A,$$

and $$QS \cot x = OQ = OP \cos b.$$

Hence $$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A,$$

which is formula **C**.

10. *Right-angled and quadrantal triangles.*

When one of the spherical angles is 90°, the formulae **A**, **B**, **C** and **D** assume simple forms. This is also the case when one side of a spherical triangle is 90°—the triangle is then said to be *quadrantal*. Rules have been given by Napier according to which the various simple formulae can be written down. The rules, however, impose an additional charge on the memory and it is much simpler to apply one of the main formulae **A** to **D** to the particular right-angled or quadrantal triangle concerned. The rules are as follows:

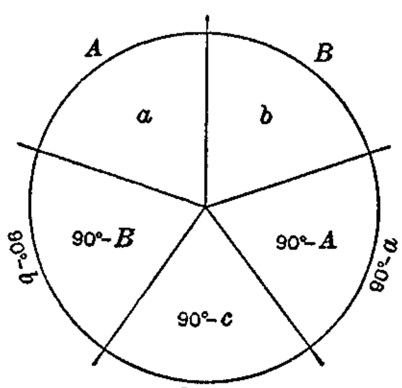

Fig. 7.

(1) Right-angled triangle in which $C = 90°$. Arrange *inside* a circle the five "circular parts" a, b, $90° - A$, $90° - c$, $90° - B$, as in Fig. 7. If any one circular part is chosen as a "middle", the two flanking parts are called "adjacents" and the two others the "opposites". The rules then are:

sin (*middle*) = product of tangents of *adjacents*;

sin (*middle*) = product of cosines of *opposites*.

(2) Quadrantal triangle in which $c = 90°$. Arrange *outside* the circle (Fig. 7) the five "circular parts" A, B, $90° - a$, $C - 90°$, $90° - b$. The two rules are then the same as for right-angled triangles.

11. *Polar formulae.*

Certain useful formulae can be obtained by means of the polar triangle which is constructed as follows (Fig. 8). Let ABC be a spherical triangle. The great circle of which BC is an arc has two poles, one in each of the hemispheres into which the sphere is divided by the great circle. Let A' be the pole in the hemisphere in which A lies. Similarly B' and C' are the appropriate poles of CA and AB. Produce BC both ways to meet $A'B'$ and $A'C'$ in L and M respectively. Then, since A' is the pole of the great circle $LBCM$, the spherical angle $B'A'C'$ (or simply A') is equal to the arc LM. Again, B' is the pole of AC, that is, the angular distance of B' from any point on AC is 90°; similarly the angular distance of A' from any point on BC is 90°. Hence the angular distance of C from B' and from A' is in each instance 90°; in other words, C is the pole of $A'B'$. Hence $CL = 90°$, and similarly $BM = 90°$. Now $LM = LB + BM = LB + 90°$. Also $BC = a$; $\therefore$ $LB = 90° - a$. Hence $A' = 180° - a$. Similarly $B' = 180° - b$ and $C' = 180° - c$. We obtain in a similar manner

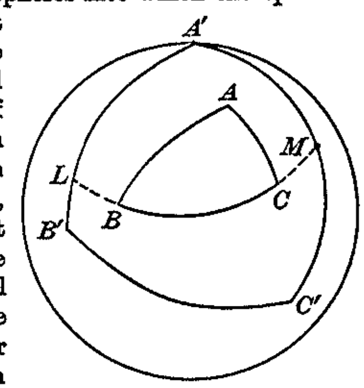

Fig. 8.

$$a' = 180° - A; \quad b' = 180° - B; \quad c' = 180° - C.$$

Now apply formula **A** to the triangle $A'B'C'$ and we have, for example,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Using the relations just found, we obtain from this equation

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

which is a formula for the triangle ABC, giving the angle A in terms of the two remaining angles and the included side. The procedure in this instance can be extended to any of the principal formulae which we have already derived, by writing $180° - a$ for A, $180° - b$ for B, etc., in the formulae **A** to **D**.

12. *Numerical example.*

To illustrate the numerical solution of a spherical triangle, we shall consider the following problem. In Fig. 9 let A and B represent two places, in north latitude, on the surface of the earth; their latitudes are respectively 24° 18′ N and 36° 47′ N, and their longitudes 133° 39′ E and 125° 24′ W respectively; it is required to find (i) the length of the great circle arc AB, (ii) the angle PAB, P being the north pole, and (iii) the most northerly point on the great circle AB.

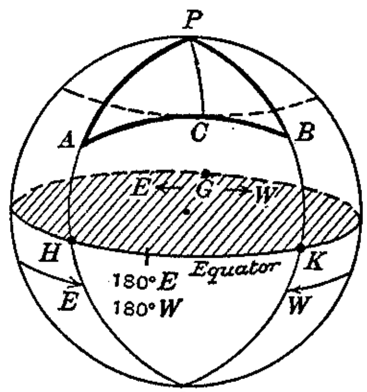

Fig. 9.

$PAHQ$ is the meridian through A cutting the equator in H. HA measures the latitude of A, i.e. $HA = 24° 18'$. PA is the colatitude of A; $\therefore PA = 90° - 24° 18' = 65° 42'$. Similarly $PB = 53° 13'$. Let the Greenwich meridian intersect the equator in G. Then, following the arrows,

$$GH = \text{long. (E) of } A = 133° 39',$$

and $$GK = \text{long. (W) of } B = 125° 24'.$$

Hence the arc $$HGK = 259° \ 3',$$

and therefore HK (the shorter of the two great circle arcs joining H and K) is 100° 57′; that is $APB = 100° 57'$. In the triangle APB we now are given the two sides PA and PB and the contained angle APB.

(i) *Calculation of AB.* By formula **A**, we have

$$\cos AB = \cos PA \cos PB + \sin PA \sin PB \cos APB,$$

which becomes, on inserting the data,

$$\cos AB = \cos 65° 42' \cos 53° 13' - \sin 65° 42' \sin 53° 13' \cos 79° 3'$$
$$\equiv \quad M \quad - N.$$

We shall use five-figure logarithms.

log cos 65° 42′·0	9·61 438	log sin 65° 42′·0	9·95 971
log cos 53° 13′·0	9·77 728	log sin 53° 13′·0	9·90 358
∴ log M =	9·39 166	log cos 79° 3′·0	9·27 864
		∴ log N =	9·14 193
∴ M =	0·24 641;	∴ N =	0·13 865.

Hence $$\cos AB \equiv M - N = 0{\cdot}10\,776;$$

$$\therefore AB = 83° 48'{\cdot}8 \equiv 5028'{\cdot}8.$$

Thus the great circle distance between A and B is 83° 48′·8 or 5028·8 nautical miles. To the nearest minute of arc, $AB = 83° 49'$.

(ii) *Calculation of PAB.* By formula **A**,

$$\cos PB = \cos AB \cos PA + \sin AB \sin PA \cos PAB.$$

In this equation, all three sides PB, AB, PA are now known and hence we can derive $P\hat{A}B$. In this instance simple geometrical considerations show that $P\hat{A}B$ is less than 90° and consequently the sine-formula **B** can be used without ambiguity; the appropriate equation is

$$\sin PAB = \frac{\sin APB . \sin PB}{\sin AB},$$

all the quantities on the right-hand side being now known. However, for purposes of illustration, we shall calculate $P\hat{A}B$ by means of formula (11). Denote AB by p, PB by a and PA by b; then

$$2s = p + a + b = 83° 49' + 53° 13' + 65° 42' = 202° 44'.$$

Hence $s = 101° 22'$; $s - p = 17° 33'$; $s - b = 35° 40'$.

In this instance, formula (11) is written

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-p)}{\sin b \sin p}}.$$

$\log\sin(s-b)$	$\equiv \log\sin$	$35° 40'$	$9{\cdot}76\,572$
$\log\sin(s-p)$	$\equiv \log\sin$	$17° 33'$	$9{\cdot}47\,934$
$\log\operatorname{cosec} b$	$\equiv \log\operatorname{cosec}$	$65° 42'$	$0{\cdot}04\,029$
$\log\operatorname{cosec} p$	$\equiv \log\operatorname{cosec}$	$83° 49'$	$0{\cdot}00\,253$

$$\therefore\ \log\sin^2\frac{A}{2} = 9{\cdot}28\,788.$$

$$\therefore\ \log\sin\frac{A}{2} = 9{\cdot}64\,394$$

$$\therefore\ \frac{A}{2} = 26° 8'$$

$$\therefore\ A = 52° 16'.$$

(iii) *Calculation of the most northerly latitude reached by the great circle AB.* Let C be the most northerly point on AB (Fig. 9). Then it is evident that the parallel of latitude through C will touch the great circle at C and that the meridian PC will be perpendicular to the great circle AB at C. Thus $P\hat{C}A$ and $P\hat{C}B$ are each 90°. In the triangle PAC, we now know PA, $P\hat{A}C$ and $P\hat{C}A$ and it is required to find PC. Clearly, formula **B** can be used; it is

$$\frac{\sin PC}{\sin P\hat{A}C} = \frac{\sin PA}{\sin P\hat{C}A},$$

and, since $P\hat{C}A = 90°$, we obtain

$$\sin PC = \sin PA \sin P\hat{A}C$$

$\log\sin PA$	$\equiv \log\sin$	$65° 42'$	$9{\cdot}95\,971$
$\log\sin P\hat{A}C$	$\equiv \log\sin$	$52° 16'$	$9{\cdot}89\,810$

$$\therefore\ \log\sin PC = 9{\cdot}85\,781$$

$$\therefore\ PC = 46° 7'.$$

Thus the latitude of C is 43° 53′.

The calculation of the longitude of C is left as an exercise to the reader.

13. *The haversine formula.*

Many calculations are appreciably shortened by the use of "haversines". The *haversine* of an angle θ (written hav θ) is defined by

$$\text{hav}\,\theta = \tfrac{1}{2}(1-\cos\theta) = \sin^2\frac{\theta}{2} \qquad \text{......(21).}$$

Since $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, we have

$$\cos\theta = 1 - 2\,\text{hav}\,\theta \qquad \text{......(22).}$$

We can now modify formula **A**, which is

$$\cos a = \cos b\cos c + \sin b \sin c \cos A.$$

According to (22) write $(1 - 2\,\text{hav}\,a)$ for $\cos a$, and $(1 - 2\,\text{hav}\,A)$ for $\cos A$. Then

$$1 - 2\,\text{hav}\,a = \cos(b-c) - 2\sin b\sin c\,\text{hav}\,A.$$

Write $1 - 2\,\text{hav}\,(b-c)$ for $\cos(b-c)$. Then we obtain

$$\mathbf{hav\,a = hav\,(b-c) + sin\,b\,sin\,c\,hav\,A} \qquad \text{......(23),}$$

which is the form of the fundamental formula expressed in terms of haversines.

From the definition in (21), hav θ is always positive and hav $(-\theta)$ = hav θ.

The haversines and log haversines of angles from 0° to 180° are found in some collections of mathematical tables among which may be mentioned *Inman's Nautical Tables* (J. D. Potter, 156 Minories, London, E. 1), which, in addition to the usual logarithmic and trigonometrical tables (to five figures), contain several other tables of astronomical value.

The calculation of the side AB (Fig. 9) by means of haversines will now be given in order to show the convenience of the method. We write (23) as follows for the triangle PAB:

$$\text{hav}\,AB = \text{hav}\,(PA - PB) + \sin PA \sin PB\,\text{hav}\,A\hat{P}B$$

$$\equiv \text{hav}\,(PA - PB) + X$$

$\log\text{hav}\,A\hat{P}B$	$\equiv \log\text{hav}$	$100° 57'$	$9{\cdot}77\,450$
$\log\sin PA$	$\equiv \log\sin$	$65° 42'$	$9{\cdot}95\,971$
$\log\sin PB$	$\equiv \log\sin$	$53° 13'$	$9{\cdot}90\,358$

$$\therefore\ \log X = 9{\cdot}63\,779$$

$$\therefore\ X = 0{\cdot}43\,430$$

$$\text{hav}\,(PA - PB) \equiv \text{hav}\,12° 29' = 0{\cdot}01\,182$$

$$\therefore\ \text{hav}\,AB = 0{\cdot}44\,612$$

$$\therefore\ AB = 83° 49',$$

which agrees with our result on p. 17.

14. *Another method.*

When two sides and the contained angle of a triangle are given, the following method is sometimes used when it is required to find the third side *and* one of the remaining angles.

To illustrate the method we shall find AB and $P\hat{A}B$ (Fig. 9). Denote AB by p, PB by a, PA by b and $A\hat{P}B$ by P. Then $a = 53° 13'$, $b = 65° 42'$ and $P = 100° 57'$.

By formulae **A**, **C** and **B**, we have

$$\cos p = \cos a \cos b + \sin a \sin b \cos P \quad \text{......(24)},$$
$$\sin p \cos A = \cos a \sin b - \sin a \cos b \cos P \quad \text{......(25)},$$
$$\sin p \sin A = \sin a \sin P \quad \text{......(26)}.$$

Define d (a positive quantity) and D by

$$\cos a = d \cos D \quad \text{......(27)},$$
$$\sin a \cos P = d \sin D \quad \text{......(28)}.$$

Hence we can write (24)–(26) as follows:

$$\cos p = d \cos (b - D) \quad \text{......(29)},$$
$$\sin p \cos A = d \sin (b - D) \quad \text{......(30)},$$
$$\sin p \sin A = \sin a \sin P \quad \text{......(31)}.$$

(i) From (27) and (28), by division,

$$\tan D = \tan a \cos P \quad \text{......(32)},$$

from which D can be calculated.

(ii) From (30) and (31),

$$\tan A = \frac{\sin a \sin P}{d \sin (b - D)},$$

which, by inserting the value of d given by (28), becomes

$$\tan A = \tan P \sin D \operatorname{cosec} (b - D) \quad \text{......(33)},$$

from which A can be calculated.

(iii) From (29) and (30),

$$\tan p = \tan (b - D) \sec A \quad \text{......(34)},$$

from which p can be calculated.

The calculations.

(i)

$\log \tan a$	$\equiv \log \tan$	$53° 13'$	0·12 631
$\log \cos P$	$\equiv \log \cos$	$100° 57'$	9·27 864 n
		$\therefore \log \tan D =$	9·40 495 n

$\cos P$ is *negative* and we attach the letter n to its logarithm to remind us of this fact. It follows that $\tan D$ is *negative*. We have assumed in formulae (27) and (28) that d is a positive quantity. Then, from the given values of a and P, it follows that

$\cos D$ is positive and $\sin D$ is negative; thus D is in the fourth quadrant, and from the value of $\log \tan D$ which we have found we obtain

$$D = 360° - 14° 15'{\cdot}6 = 345° 44'{\cdot}4.$$

Hence

$$b - D \equiv 65° 42' - 345° 44'{\cdot}4 = -280° 2'{\cdot}4 = 79° 57'{\cdot}6.$$

(ii)

$\log \tan P$	$\equiv \log \tan$	$100° 57'$	0·71 338 n
$\log \sin D$	$\equiv \log \sin$	$345° 44'{\cdot}4$	9·39 151 n
$\log \operatorname{cosec} (b - D)$	$\equiv \log \operatorname{cosec}$	$79° 57'{\cdot}6$	0·00 670
		$\therefore \log \tan A =$	0·11 159

and, as A is less than 180°, we have

$$P\hat{A}B \equiv A = 52° 16'{\cdot}9.$$

(iii)

$\log \tan (b - D)$	$\equiv \log \tan$	$79° 57'{\cdot}6$	0·75 192
$\log \sec A$	$\equiv \log \sec$	$52° 16'{\cdot}9$	0·21 340
		$\therefore \log \tan p =$	0·96 532

$$\therefore AB \equiv p = 83° 49',$$

agreeing with the previous calculations of AB.

15. *The trigonometrical ratios for small angles.*

If θ is a small angle and *expressed in circular measure*, we have the well-known approximate formulae:

$$\sin \theta = \theta \text{ radians}; \quad \cos \theta = 1; \quad \tan \theta = \theta \text{ radians} \quad \text{......(35)}.$$

Now

$$1 \text{ radian} = 57° 17' 45''$$
$$= 3437\tfrac{3}{4}'$$
$$= 206265'',$$

so that

$$1'' = \frac{1}{206265} \text{ radian},$$

and

$$1' = \frac{1}{3438} \text{ radian, approximately.}$$

Hence, by the first equation of (35), when θ is successively 1″ and 1′,

$$\sin 1'' = \frac{1}{206265} \quad \text{......(36)},$$

and

$$\sin 1' = \frac{1}{3438} \quad \text{......(37)}.$$

If θ'' denotes the *number* of seconds of arc in θ radians, then $\theta = \dfrac{\theta''}{206265}$ and consequently

$$\sin\theta = \frac{\theta''}{206265},$$

which may be written

$$\sin\theta'' = \theta'' \sin 1'' \qquad \text{......(38).}$$

Similarly,

$$\sin\theta' = \theta' \sin 1' \qquad \text{......(39),}$$

where θ' is expressed in minutes of arc.

In a similar way, we find

$$\tan\theta'' = \theta'' \sin 1''.$$

In spherical astronomy, certain angles are frequently expressed in terms of hours, minutes and seconds of time, according to the following relations:

$$24 \text{ hours} = 360°;\ 1^h = 15°;\ 1^m = 15' \text{ and } 1^s = 15'' \qquad \text{......(40).}$$

Thus we obtain the approximate formulae

$$\sin 1^m = \sin 15' = 15 \sin 1' \qquad \text{......(41),}$$

$$\sin 1^s = \sin 15'' = 15 \sin 1'' \qquad \text{......(42).}$$

If H is a small angle, which, when expressed in minutes of time, will be denoted by H^m, then

$$\sin H = H^m \sin 1^m = 15H^m \sin 1' \qquad \text{......(43).}$$

Similarly, if we express H in terms of seconds of time, we have

$$\sin H = H^s \sin 1^s = 15H^s \sin 1'' \qquad \text{......(44).}$$

These results will be of use in subsequent chapters.

16. *Delambre's and Napier's analogies.*

For reference, we give the following formulae, originally due to Delambre, and known as Delambre's analogies:

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) = \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b) \text{......(45),}$$

$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b) \text{......(46),}$$

$$\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b) \text{......(47),}$$

$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b) \text{......(48).}$$

These formulae are easily derived from the principal formulae already discussed in the previous pages.

Taking these equations in pairs, we obtain Napier's analogies:

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c \qquad \text{......(49),}$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c \qquad \text{......(50),}$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C \qquad \text{......(51),}$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C \qquad \text{......(52).}$$

EXERCISES

1. In the spherical triangle ABC, $C = 90°$, $a = 119° 46' 36''$ and $B = 52° 25' 38''$. Calculate the values of b, c and A.

[Ans. 48° 26′ 49″, 109° 14′ 0″ and 113° 10′ 46″.]

2. In the triangle ABC, $a = 57° 22' 11''$, $b = 72° 12' 19''$ and $C = 94° 1' 49''$. Calculate the values of c, A and B.

[Ans. 83° 46′ 32″, 57° 40′ 45″ and 72° 49′ 50″.]

3. In the triangle ABC, $c = 90°$, $B = 62° 20' 42$ and $a = 136° 19' 0''$. Calculate the values of A, C and b.

[Ans. 139° 46′ 13″, 69° 14′ 45″ and 71° 18′ 9″.]

4. Two ships X and Y are steaming along the parallels of latitude 48° N and 15° S respectively, in such a way that at any given moment the two ships are on the same meridian of longitude. If the speed of X is 15 knots,* find the speed of Y.

5. A and B are two places on the earth's surface with the same latitude ϕ; the difference of longitude between A and B is $2l$. Prove that (i) the highest latitude reached by the great circle AB is $\tan^{-1}(\tan\phi \sec l)$, and (ii) the distance measured along the parallel of latitude between A and B exceeds the great circle distance AB by

$$2 \operatorname{cosec} 1' \left[l \cos\phi - \sin^{-1}(\sin l \cos\phi)\right] \text{ nautical miles.}$$

6. The most southerly latitude reached by the great circle joining a place A on the equator to a place B in south latitude ϕ is ϕ_1. Prove that the difference of longitude between A and B is $90° + \cos^{-1}(\tan\phi \cot\phi_1)$.

7. The positions of A and B are respectively: Lat. 39° 20′ S, Long. 110° 10′ E and Lat. 44° 30′ S, Long. 46° 20′ W. Show that, if a ship steams from A to B by the shortest possible route without crossing the parallel of 62° S, the distance steamed is 5847·6 nautical miles.

* The *knot* is the unit of speed in use at sea; it is 1 nautical mile per hour.