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The Contribution of High Order Zernike Modes to Wavefront Tilt.

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ABSTRACT

The contribution of high order Zernike modes on average phase gradient is investigated. The resulting temporal power spectra are compared to data collected using the tip/tilt servo at the Sydney University Stellar Interferometer (SUSI). The low frequency part fo the power spectrum follows a $-\frac{2}{3}$ power law which is in agreement with other similar work. Due to the effect of higher order terms, the high frequency power law for average phase gradient becomes $-\frac{11}{3}$ rather than the $-\frac{17}{3}$ predicted for the 'tilt' component of the Zernike representation of the wavefront.

1. Introduction

Both for the design of tip/tilt servos and for measuring the effects of astronomical seeing it is important to know the power spectra of the phase gradient across the aperture. One convenient way of describing aberrations in the phase across a circular aperture are the polynomials of Zernike. Propagation through a turbulent atmosphere is often expressed in terms of the effect on the Zernike coefficients.

As will be shown below, it is possible to express the average phase gradient in terms of the sum of higher order Zernike modes weighted by their value at the aperture edge. Based on this expansion a form for the temporal power spectrum of the phase gradient is derived and compared with a spectrum measured using the tip/tilt servo of the Sydney University Stellar Interferometer.

2. Zernike Polynomials

Zernike polynomials are a normalized set of orthogonal functions defined on a unit circle. There are many forms for these polynomials each using a different normalization. We shall use the nomenclature set out by Noll (Noll 1976). More detail concerning the polynomials of Zernike can be found in other texts (Born and Wolf 1987, Wang and Markey 1978, Tyson 1992) while methods for experimental determination are set out by Lawrence and Chow (Lawrence and Chow 1984).

Using polar coordinates ρ and θ, normalized for an unobscured aperture of radius R, the phase of the wavefront across the aperture can be written

$$\varphi(R\rho,\theta) = \sum_j a_j Z_j(\rho,\theta) \tag{1}$$

where $Z_j(\rho,\theta)$ is the Zernike polynomial of order j, a_j are the expansion coefficients given by

$$a_j = \int d\boldsymbol{\rho}\, W(\rho)\, \varphi(R\rho,\theta)\, Z_j(\rho,\theta) \tag{2}$$

and the weighting function

$$W(\rho) = \begin{cases} 1/\pi & \rho \leq 1 \\ 0 & \rho > 1 \end{cases} \tag{3}$$

is added so that the integral can be taken over all space. The Zernike polynomials themselves are given by

$$Z_j(\rho,\theta) = \sqrt{n+1}\, R_n^m(\rho) \times \begin{cases} \sqrt{2}\cos m\theta & m \neq 0, \quad j \text{ even} \\ \sqrt{2}\sin m\theta & m \neq 0, \quad j \text{ odd} \\ 1 & m = 0 \end{cases} \tag{4}$$

where

$$R_n^m(\rho) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!}{s!\,[(n+m)/2-s]!\,[(n-m)/2-s]!}\, \rho^{n-2s}. \tag{5}$$

The constants m and n are integers such that $m \leq n$ and $n - |m|$ is even. The index j is used to order the modes.

Zernike polynomials follow the orthogonality relation

$$\int d\mathbf{r} W(r) Z_j(\mathbf{r}) Z_{j'}(\mathbf{r}) = \delta_{jj'}. \tag{6}$$

The only other property of the Zernike polynomials that will be required for an analysis of turbulence is their Fourier transform $Q_j(k,\phi)$ defined by

$$W(\rho) Z_j(\rho,\theta) = \int d\,\mathbf{k} Q_j(k,\phi) \exp(-2\pi i \mathbf{k}\cdot\boldsymbol{\rho}) \tag{7}$$

and written as (Born and Wolf 1987)

$$Q_j(k,\phi) = \sqrt{n+1}\, \frac{J_{n+1}(2\pi k)}{\pi k} \times \begin{cases} (-1)^{(n-m)/2}\, i^m \sqrt{2}\cos m\phi & m \neq 0, \text{even}\, j \\ (-1)^{(n-m)/2}\, i^m \sqrt{2}\sin m\phi & m \neq 0, \text{odd}\, j \\ (-1)^{n/2} & m = 0 \end{cases} \tag{8}$$

3. Temporal Power Spectra of Zernike Coefficients

The paper by Roddier et al (Roddier et al 1993) gives a derivation of the form of the temporal power spectra of Zernike coefficients due to Kolmogorov turbulence. Basically, they show that spatial power spectrum of each coefficient is given by

$$W_{a_j}(\mathbf{k}) = W_\phi(\mathbf{k}) \cdot W_z(\mathbf{k}) \tag{9}$$

where $W_\phi(\mathbf{k})$ is the spatial power spectrum of the phase at ground level and $W_z(\mathbf{k})$ is the squared modulus of the Fourier Transform of the the polynomial. The temporal power spectrum is then given by

$$\Phi_{a_j}(f) = \frac{R}{v} \int W_{a_j}(\frac{fR}{v}, k_y) dk_y \tag{10}$$

where v is the wind speed, assumed to be in the same direction as the $\phi = 0$ axis, the frequency is defined as $f = vk_x/R$ and (k_x, k_y) are the Cartesian components of $\mathbf{k}$. From Noll (Noll 1976) we obtain

$$W_\phi(\mathbf{k}) = 0.023 \left(\frac{R}{r_0}\right)^{5/3} k^{-11/3}. \tag{11}$$

Combining these results with equation (8) we get

$$\begin{aligned} \Phi_{a_j}(f) &= \frac{0.023}{\pi^2}\,(n+1)\,\left(\frac{R}{r_0}\right)^{5/3}\,\frac{R}{v}\,\int dk_y \left(\frac{f^2R^2}{v^2} + k_y^2\right)^{-17/6} \left| J_{n+1}\left(2\pi\sqrt{\frac{f^2R^2}{v^2} + k_y^2}\right)\right|^2 \\ &\times \begin{cases} 2\,\cos^2 m\phi & m \neq 0, \text{even } j \\ 2\,\sin^2 m\phi & m \neq 0, \text{odd } j \\ 1 & m = 0 \end{cases} \end{aligned} \tag{12}$$

which is the same as equation (16) in the Roddier et al (Roddier et al 1993) paper except that the constant of proportionality has been added and a minor typographical error has been corrected.

4. Zernike Polynomial Expansion of Wavefront Tilt

It is common to use the second and third modes of the Zernike polynomials to model tilt. As shall be shown in section (5), this works well at low frequencies. At high frequencies higher order modes will have a significant effect.

An alternative definition of wavefront tilt is the average phase gradient across the aperture defined

$$\overline{\frac{\partial\varphi(R\rho,\theta)}{\partial X}} = \int d\boldsymbol{\rho} W(\rho)\frac{\partial\varphi(R\rho,\theta)}{\partial X} \tag{13}$$

where X is an axis at an angle ϵ to the x axis such that

$$X = \rho\cos(\theta - \epsilon). \tag{14}$$

Wavefront tilt can be obtained from average phase gradient via

$$\Theta_{\text{tilt}} = \frac{\lambda}{2\pi R}\,\overline{\frac{\partial\varphi(R\rho,\theta)}{\partial X}}. \tag{15}$$

Given that we know the Zernike representation of a wavefront we can find the average phase gradient across the aperture. By substituting equation (1) into equation (13) and reversing the orders of summation and integration the problem becomes one of summing terms of the form

$$\overline{\frac{\partial Z_j(\rho,\theta)}{\partial X}} = \int d\boldsymbol{\rho} W(\rho)\frac{\partial Z_j(\rho,\theta)}{\partial X}. \tag{16}$$

If we first consider the partial differential

$$\frac{\partial Z_j(\rho,\theta)}{\partial X} = \frac{\partial Z_j(\rho,\theta)}{\partial \rho}\frac{\partial \rho}{\partial X} + \frac{\partial Z_j(\rho,\theta)}{\partial \theta}\frac{\partial \theta}{\partial X} \tag{17}$$

$$\begin{aligned} &= \sqrt{n+1}\,\frac{\partial R_n^m(\rho)}{\partial \rho}\left[\cos\theta\cos\epsilon - \sin\theta\sin\epsilon\right] \times \begin{cases} \sqrt{2}\cos m\theta & m \neq 0, \quad j \text{ even} \\ \sqrt{2}\sin m\theta & m \neq 0, \quad j \text{ odd} \\ 1 & m = 0 \end{cases} \\ &+ \sqrt{n+1}\,\frac{R_n^m(\rho)}{\rho}\left[\cos\theta\sin\epsilon - \sin\theta\cos\epsilon\right] \times \begin{cases} -\sqrt{2}\sin m\theta & m \neq 0, \quad j \text{ even} \\ \sqrt{2}\cos m\theta & m \neq 0, \quad j \text{ odd} \\ 1 & m = 0 \end{cases} \end{aligned} \tag{18}$$

it is clear that for the $m = 0$ modes the integral across the aperture will be zero due to the $\sin\theta$ and $\cos\theta$ terms. Furthermore, due to the standard integrals

$$\int_0^{2\pi} \cos m\theta \sin\theta d\theta = \int_0^{2\pi} \sin m\theta \cos\theta d\theta = 0 \tag{19}$$

and

$$\int_0^{2\pi} \cos m\theta \cos\theta d\theta = \int_0^{2\pi} \sin m\theta \sin\theta d\theta = \begin{cases} \pi & m = 1 \\ 0 & m \neq 1, \end{cases} \tag{20}$$

all other modes go to zero when integrated over the aperture except those for which $m = 1$. If one performs the integration for these modes it is relatively easy to show that

$$\overline{\frac{\partial Z_j(\rho,\theta)}{\partial X}} = \begin{cases} Z_j(1,\epsilon) & m = 1 \\ 0 & m \neq 1 \end{cases} \tag{21}$$

resulting in the final expression for average phase gradient

$$\Theta_{\text{tilt}}(\epsilon) = \sum_{m=1} a_j(t) Z_j(1,\epsilon). \tag{22}$$

5. Phase Gradient Power Spectrum

With the expression for average phase gradient in the direction ϵ given in equation (22) and the expression for the Zernike coefficient power spectra given in equation (12) we are in a position to find an expression for the power spectrum of average phase gradient. While the higher order modes are not completely independent, the cross terms are either zero or small (Noll 1976, Roddier 1990) and we can approximate the phase gradient power spectrum by

$$\Phi_{\Theta_{\text{tilt}}}(f) = \sum_{m=1} Z_j^2(1,\epsilon)\Phi_{a_j}(f). \tag{23}$$

As the example is in figure (1) shows, as each higher order mode is added to the calculation the

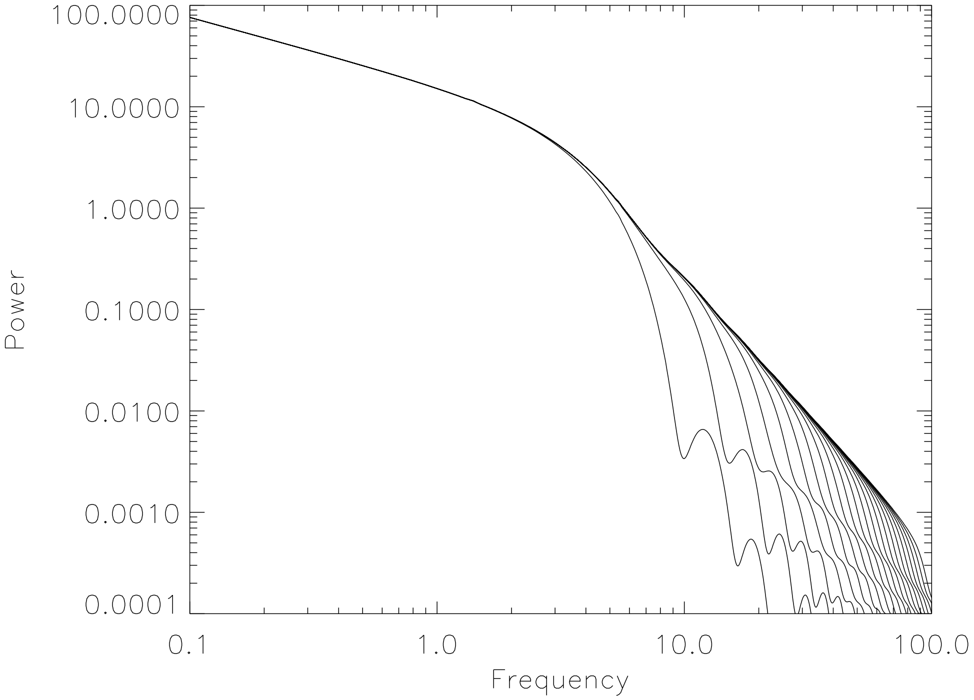

Fig. 1.— Theoretical predictions for the average phase gradient power spectrum with frequency expressed as a multiple of f_0. The lowest plot is the same as the tilt component in the previous figure. Each plot above includes higher and higher order corrections up to and including $j = 1000$. Note how each correction contributes a change to higher frequencies. Taken altogether, the low frequency part is not changed while the slope of the high frequency part is changed dramatically.

slope of the high frequency part is modified. The total effect is to create a slope of $-\frac{11}{3}$. The low frequency part does not change from the $-\frac{2}{3}$ power law.

Figure (2) shows some data collected using the tip/tilt servo at SUSI (ten Brummelaar and Tango 1994) while tracking α Car for five minutes with a bandwidth of 150Hz and an aperture size of 10.5cm along with a least squares fit of equation (23) including all modes up to $j = 1000$. This fit resulted in $r_0 = 7.73 \overset{\times}{\div} 0.20$ cm and $v_\perp = 3.66 \pm 0.10\mathrm{ms}^{-1}$ for a wavelength of 440 nm, values typical of the site. Note that equation (12) is computationally expensive and these fits can take a long time to calculate. An approximate form for the coefficient temporal power spectra (see appendix A) was used to get initial values for this fitting process thereby reducing the calculation times substantially.

The $-\frac{2}{3}$ power law at low frequencies fits well, as also confirmed experimentally by others (Nightingale and Buscher 1991, Doel et al 1990, Colavita 1987, Acton et al 1992). The high frequency power law of $-\frac{11}{3}$ does not fit as well, however, it is likely that instrument roll off is occurring at high frequencies causing a steeper slope than that predicted. Acton et al (Acton et al 1992), using much larger bandwidths, also find a steeper slope at frequencies greater than 150Hz which they contribute to a large inner scale length. Inspection of figure (1) shows that at very high frequencies the contribution of high order terms diminishes leaving a final slope of $-\frac{17}{3}$ which may explain the results of Acton et al (Acton et al 1992)although these measurements were performed during the day and may simply reflect the difference between day and night-time seeing.

6. Conclusion

The temporal power spectrum of average phase gradient across an aperture due to Kolmogorov turbulence was calculated based on Zernike polynomial expansion of average phase gradient. The calculation gives a low frequency power law of $-\frac{2}{3}$ and a high frequency power law of $-\frac{11}{3}$ rather than the $-\frac{17}{3}$ cited for the tilt component of a Zernike expansion. The $-\frac{11}{3}$ slope is in good agreement with past predictions and experimental evidence.

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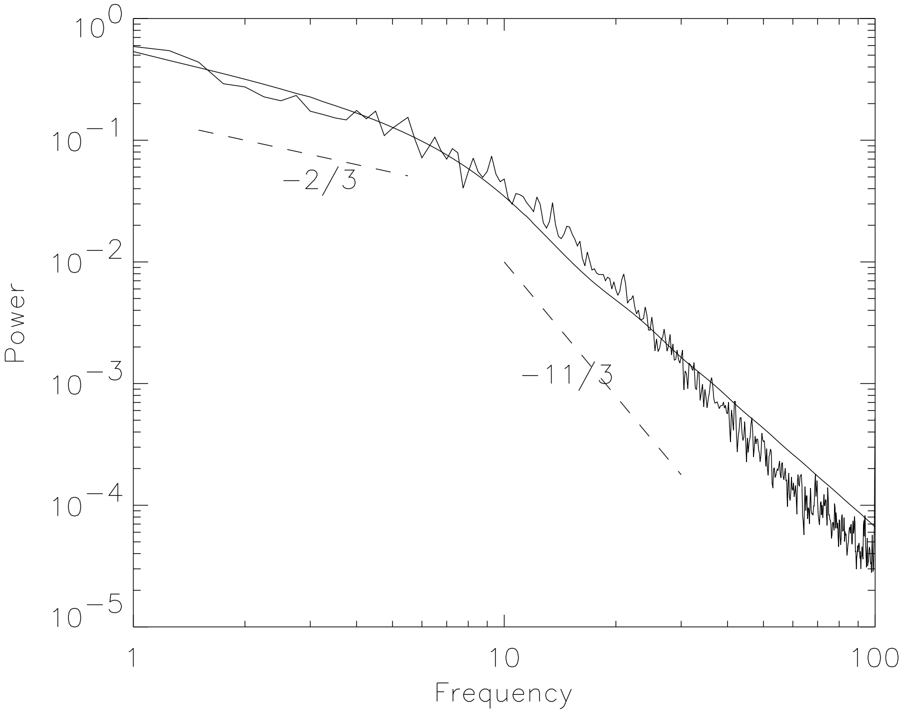

Fig. 2.— The power spectrum of tilt as measured by the tilt correction system at SUSI, together with a fit of the theoretical spectrum and some dashed lines representing simple power laws. The system was tracking the star α Car with a bandwidth of 150Hz. The frequency is measured in Hertz.

A.

We now derive an approximate form for the temporal power spectra of Zernike coefficients due to Kolmogorov turbulence. Combining the definition of the expansion coefficients given in equation (2) and explicitly adding the time dependence of the phase across the aperture, the temporal autocorrelation of a Zernike coefficient can be written in the form

$$C_{a_j}(\tau) = \int d\boldsymbol{\rho} \int d\boldsymbol{\rho}' \, W(\rho) Z_j(\boldsymbol{\rho}) \, C_\varphi(R\boldsymbol{\rho}, R\boldsymbol{\rho}', \tau) \, W(\rho') Z_j(\boldsymbol{\rho}') \tag{A1}$$

where the integral contains the autocorrelation function of phase

$$C_\varphi(R\boldsymbol{\rho}, R\boldsymbol{\rho}', \tau) = \langle \varphi(R\boldsymbol{\rho}, t) \varphi(R\boldsymbol{\rho}', t+\tau) \rangle. \tag{A2}$$

Using the power law (Bracewell 1986) on both the $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ variables, equation (A1) can be written in Fourier space as

$$C_{a_j}(\tau) = \int d\mathbf{k} \int d\mathbf{k}' \, Q_j^*(\mathbf{k}) \, \Phi(\mathbf{k}, \mathbf{k}', \tau) \, Q_j(\mathbf{k}') \tag{A3}$$

where $\Phi(\mathbf{k}, \mathbf{k}', \tau)$ is the spatial Fourier transform of $C_\varphi(R\boldsymbol{\rho}, R\boldsymbol{\rho}', \tau)$ with respect to both $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$. From Noll (Noll 1976) we obtain

$$\mathcal{FT}\left[C_\varphi(\boldsymbol{\rho}, \boldsymbol{\rho}', 0)\right] = \Phi_\varphi(\boldsymbol{\kappa}) \delta(\mathbf{k} - \mathbf{k}') = 0.023 \left(\frac{1}{r_0}\right)^{\frac{5}{3}} k^{-\frac{11}{3}} \delta(\mathbf{k} - \mathbf{k}'). \tag{A4}$$

This equation is a direct consequence of equation (11) and the auto-correlation theorem (Bracewell 1986). If the two spatial wave numbers $\mathbf{k}$ and $\mathbf{k}'$ have the same value, $C_\varphi(\boldsymbol{\rho}, \boldsymbol{\rho}', 0)$ is the spatial autocorrelation of the phase across the aperture whose Fourier transform is the spatial power spectrum given by equation (11). If the spatial wave numbers are not equal, turbulence theory gives us no information about $C_\varphi(\boldsymbol{\rho}, \boldsymbol{\rho}', 0)$ and we therefore introduce the delta function shown in equation (A4). Any other model for turbulence could be used at this point as long as it supplies an expression for the spatial power spectrum for phase at ground level.

By the similarity theorem (Bracewell 1986) we find that

$$\mathcal{FT}\left[C_\varphi(R\boldsymbol{\rho}, R\boldsymbol{\rho}', 0)\right] = 0.023 \, (\frac{R}{r_0})^{\frac{5}{3}} \, k^{-\frac{11}{3}} \, \delta(\mathbf{k} - \mathbf{k}'). \tag{A5}$$

We now invoke the Taylor hypothesis by assuming that the temporal autocorrelation function is related directly to the spatial autocorrelation. It is at this point that an approximation is made and we shall treat the average wind speed perpendicular to the optical axis $v_\perp$ as a scalar rather than a vector. While wind velocity is not a scalar, standard turbulence theory assumes local isotropy and it is not uncommon to use such an approximation (see for example the paper by Tango and Twiss (1980)). We therefore write

$$\langle \varphi(R\rho, t) \varphi(R\rho', t+\tau) \rangle = \langle \varphi(R\rho, t) \varphi(R(\rho' - v_\perp \tau / R), t) \rangle. \tag{A6}$$

By using the shift theorem of Fourier transforms (Bracewell 1986) we can now write the required transform

$$\Phi(\mathbf{k},\mathbf{k}',\tau) = 0.023 \left(\frac{R}{r_0}\right)^{\frac{5}{3}} \exp(-2\pi i v_\perp \tau k/R)\, k^{-\frac{11}{3}}\, \delta(\mathbf{k}-\mathbf{k}'). \tag{A7}$$

Combining equations (8), (A3) and (A7) we arrive at an expression for the temporal autocorrelation function of the jth mode of the Zernike polynomial expansion of the wavefront across the aperture due to Kolmogorov turbulence. Due to the delta function in equation (A7) the integral over $\mathbf{k}'$ becomes trivial, as is the integral over the angular part. The final expression becomes

$$C_{a_j}(\tau) = \frac{0.046}{\pi} \left(\frac{R}{r_0}\right)^{\frac{5}{3}} \int dk\, \exp(-2\pi i v_\perp \tau k/R)\, k^{-\frac{8}{3}}\, \frac{J^2_{n+1}(2\pi k)}{k^2}. \tag{A8}$$

It remains to perform a Fourier transform of this expression to find the power spectrum of the selected mode. As the only part of equation (A8) containing a time dependence is the periodic exponential function, by swapping the order of integration we need only consider the transform of this part. Noting further that the power spectrum must be real and is undefined for negative frequencies, we write

$$\begin{aligned} \textstyle\int df \exp(-2\pi i v_\perp \tau k/R) \exp(-2\pi i f \tau) &= \delta(f - v_\perp k/R) \\ &= R/v_\perp \delta(k - Rf/v_\perp). \end{aligned} \tag{A9}$$

Combining equations (A8) and (A9) and using the autocorrelation theorem we find that

$$\Phi_{a_j}(f) = 0.096\, (2\pi)^{\frac{11}{3}} \left(\frac{R}{r_0}\right)^{\frac{5}{3}} \frac{R}{v_\perp}\, (n+1) \left(\frac{f}{f_0}\right)^{-\frac{8}{3}} \frac{J^2_{n+1}(f/f_0)}{(f/f_0)^2} \tag{A10}$$

where $f_0 = v_\perp/(2\pi R)$.

In order to test this result we calculate the variance and compare the result to those previous published. The variance of any Zernike coefficient is given by the total power in its spectrum,

$$\sigma^2_{a_j} = \int_0^\infty df \Phi_{a_j}(f). \tag{A11}$$

Substituting equation (A10) and using the variable change $k = Rf/v_\perp$ the variance of any given Zernike coefficient will be

$$\sigma^2_{a_j} = \frac{0.046}{\pi} \left(\frac{R}{r_0}\right)^{\frac{5}{3}} (n+1) \int_0^\infty dk\, k^{-\frac{8}{3}}\, \frac{J^2_{n+1}(2\pi k)}{k^2}. \tag{A12}$$

This is in exact agreement with the more general expression for the covariance of the expansion coefficients derived by Noll (Noll 1976) and later corrected by Wang and Markey (Wang and Markey 1980):

$$\langle a_j a_{j'} \rangle = \begin{cases} \frac{0.046}{\pi} \left(\frac{R}{r_0}\right)^{\frac{5}{3}} \sqrt{(n+1)(n'+1)} & \\ \quad \times (-1)^{(n+n'-2m)/2} \delta_{mm'} & \\ \quad \times \int dk k^{-\frac{8}{3}} \frac{J^2_{n+1}(2\pi k)}{k^2} & \text{for } (j-j') \text{ even} \\ 0 & \text{for } (j-j') \text{ odd} \end{cases} \tag{A13}$$

which will reduce to equation (A12) when $j = j'$.

The form for the temporal power spectra of Zernike coefficients derived above does not require numerical quadrature to calculate and can therefore be used in numerically intensive simulations and calculations. It does not correctly predict the low frequency part of the spectra but the 'turn over' frequency, high frequency $-\frac{17}{3}$ power law and total power, or variance, are preserved.

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