ENERGY GENERATION

Gravitational Energy

Using $4\pi I_{11} = G\int_0^R \frac{m(r)\rho 4\pi r^2}{r}dr$ = Total grav. energy and limits derived earlier, $\Omega \leq -\frac{3}{5}\frac{GM^2}{R}$ ~ uniform density sphere

For a polytrope we've already shown: $\Omega = \frac{-3}{5-n}\frac{GM^2}{R}$, which is more negative than uniform, thanks to extra concentration.

We can write the stellar VT, w/ no mass motions ($T=0$) as:

$$2\mathcal{T} + \Omega = 0$$

But $d\mathcal{T} = \underbrace{\tfrac{3}{2}RT\,dm}_{\text{int. KE density}} = \frac{3}{2}(C_p - C_v)T\,dm$ since $C_p = C_v + R$

and $\left.\frac{dQ}{dT}\right|_\alpha = C_\alpha \Rightarrow$ the int. heat energy,

$dU = C_v T\,dm$. $\therefore d\mathcal{T} = \frac{3}{2}\left(\frac{C_p - C_v}{C_v}\right)dU$ Recalling $\gamma = \frac{C_p}{C_v}$,

$$\boxed{\mathcal{T} = \frac{3}{2}\langle\gamma - 1\rangle U}$$

total int KE ($\mathcal{T}$), total heat en. (U)

For monatomic: $\gamma = 5/3$ so

$$\mathcal{T} = U$$

Go into Virial Thm, w/ assumption γ = const $\Rightarrow$

$$U = \frac{-\Omega}{3(\gamma-1)}$$

$$E = U + \Omega = -(3\gamma - 4)U$$

$$E = \frac{3\gamma-4}{3(\gamma-1)}\Omega$$

Stability requires $E < 0 \Leftrightarrow \gamma > 4/3 \Leftrightarrow n < 3$ for a polytrope.

Rotational Energy Can't Be Too Large:

$T_{rot} = \frac{1}{2} I \omega^2 \quad I < \frac{2}{5} M R^2$ (uniform sphere)

Support against centrifugal force $\Rightarrow$

$\frac{GM}{R^2} > \omega^2 R$ or $\omega^2 < \frac{GM}{R^3}$

If centrally condensed, this becomes $\omega^2 < \frac{8GM}{27R^3}$

and combining we see:

$T_{rot} = \frac{1}{2} I \omega^2 < \frac{8GM^2}{135R} \lesssim 0.1\,\Omega$

Timescales Redux! $\tau_d \Rightarrow \frac{d^2 I}{dt^2} \simeq -\Omega \simeq \frac{I}{\tau_d^2}$

or $\tau_d \simeq \frac{\frac{2}{5} M R^2}{3/5\, GM^2/R} \Rightarrow \tau_d = \left(\frac{2R^3}{3GM}\right)^{1/2}$ for dynamical t.s.

Actual free-fall: $\frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2}$ Tricky, but can $\int$ directly:

$\tau_{ff} = \frac{\pi}{2}\left(\frac{R^3}{2GM}\right)^{1/2} \simeq 1.36\, \tau_d = \left(\frac{3\pi}{32G\langle\rho\rangle}\right)^{1/2}$

Sound crossing time: $\tau_s = \frac{R}{\langle c_s \rangle} = R\left\langle\frac{\gamma P}{\rho}\right\rangle^{-1/2} = R\left\langle\left(\frac{\gamma k T}{\mu m_H}\right)^{-1/2}\right\rangle$

If $\gamma = 5/3$: $\langle c_s \rangle = \left(\frac{5k}{3\mu m_H}\right)^{1/2} \langle T^{1/2} \rangle \simeq 3\times10^3\,\mathrm{cm\,s^{-1}}\,T^{1/2}$

Can use T estimates to find: $\tau_s = \left(\frac{3R^3}{GM}\right)^{1/2} \simeq 2.12\, \tau_d$

All of these are $\approx \left(\frac{R^3}{GM}\right)^{1/2}$ or $\langle\rho\rangle^{-1/2}$ (~27 min for $\odot$).

Kelvin-Helmholtz: Slow contraction, so $\langle T \rangle \simeq 0$

$\therefore \frac{1}{2}\langle\Omega\rangle = -\langle U \rangle$ as discussed before; half can be radiated away, so: $\tau_{K\text{-}H} = \frac{-\Omega}{L} = \frac{\frac{3}{5} GM^2}{RL}$. Let τ_{th} = time needed for L to get rid of internal heat en

$\tau_{th} = \frac{\langle U \rangle}{L} = \frac{-\Omega}{\langle 3(\gamma-1)\rangle L} \simeq \frac{\tau_{K\text{-}H}}{\langle 3(\gamma-1)\rangle}$ — of same order & ~10^{11} longer than τ_d.

Nuclear: $\tau_n = \frac{\chi_n M c^2}{L}$ w/ χ_n = fraction available for conversion:

Except in late (pre SN) stages:

$\tau_d \simeq \tau_{ff} \lesssim \tau_s \ll \tau_{KH} \simeq \tau_{th} \ll \tau_n$

NUCLEAR ENERGY GENERATION

Nuclear Reactions: Kinematics

Elastic Scattering $a=b,\ X=Y$ | Inelastic Scatt $a=b^*,\ Y=X^*$ | Photoemission $b=\gamma$

In the following full reactions (in two notations)

$$a + X \to Y + b \qquad \text{i.e. } d + {}^{12}C \to {}^{13}C + p$$

$$X(a,b)Y \qquad \text{i.e. } {}^{12}C(d,p){}^{13}C$$

Compound nucleus: $X + a \to C^* \to Y + b$ with a delay.

Stick to NR collisions since $T_\star \ll 6\times10^9\,K$ for most $\star$s so even e's have $\Gamma \sim 1$.

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{V}_{cm} \qquad (1)$$

$$\therefore \vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \qquad \text{w.r.t. CM:}$$

$$m_1(\vec{v}_1 - \vec{V}_{cm}) = \mu\vec{v} \qquad \text{w/ } \mu = \frac{m_1 m_2}{m_1+m_2} \qquad \vec{v} = \vec{v}_1 - \vec{v}_2 \qquad (2)$$
$$m_2(\vec{v}_2 - \vec{V}_{cm}) = -\mu\vec{v}$$

Initial KE: $KE_i = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$

$$= \frac{1}{2}(m_1+m_2)V_{cm}^2 + \underbrace{\frac{1}{2}\mu v^2}_{\text{KE in the CM system — available for changing things.}}$$

$KE_i = KE_f$ except that $\Delta KE = -\Delta M c^2$, $\therefore$ can add energy!
Negligible energy is usually added to CM motion, so if:

E_{aX} is CM KE of a & X:

$$E_{aX} + (M_a + M_X)c^2 = E_{bY} + (M_b + M_Y)c^2 \qquad (3)$$

Either nuclear or atomic masses could be used since charges are conserved & # of e's match # of p's. All nuclear reactions conserve:

1) Baryon # 2) Charge 3) Lepton # 4) Energy-momentum

Atomic mass excess in units of energy are:

$$\Delta M_{AZ} = (M_{AZ} - A\underbrace{M_u}_{1\,AMU = \frac{1}{12}\,mass(^{12}C)})c^2 = [M_{AZ}(amu) - A]c^2 M_u \qquad (4)$$

or $\Delta M_{AZ} = 931.478\,(M_{AZ} - A)$ MeV

$\therefore (3) \Rightarrow E_{aX} + (\Delta M_a + \Delta M_X) = E_{bY} + (\Delta M_b + \Delta M_Y)$ [w/ these in energy units]

A table of mass excesses for each isotope gives diff. bet. mass & AMU & enables am't of energy/reaction to be determined.

I.e. $^{12}C(d,p)^{13}C$: $E_{d,^{12}C} + 13.13591 + \overset{^{12}C}{0} = E_{p,^{13}C} + 7.28899 + 3.12460$

$$\Longrightarrow E_{p,^{13}C} = E_{d,^{12}C} + 2.7223 \text{ MeV}$$

In general: $Q \equiv \Delta E = (M_X + M_a - M_Y - M_b)c^2 = \Delta M_a + \Delta M_X - \Delta M_b - \Delta M_Y$

Cross-Sections $\sigma(cm^2) \equiv \dfrac{\#\,reactions/nucleus\ X/unit\ time}{number\ of\ incident\ particles/cm^2/unit\ time}$

$\therefore$ Reaction rate: $r(v) = \sigma(v)\,\underbrace{v}_{relative\ velocity}\,N_a N_X \qquad (5)$

In LTE $\exists$ a spectrum of relative velocities between types of particles a & X w/ a distribution $\phi(v)dv$, defined $\ni \int \phi(v)dv = 1$

$$\therefore (5) \Rightarrow r_{aX} = N_a N_X \int_0^\infty v\sigma(v)\phi(v)dv = N_a N_X \langle \sigma v \rangle \qquad (6)$$

Thus, the big job is computing $\langle \sigma v \rangle \equiv \lambda \sim$ reaction rate/pair of particles

Note that if the particles are identical $(a = X)$ then the # of pairs/vol^2 and, technically (# distinct pairs):

$$r_{aX} = (1 + \delta_{aX})^{-1} N_a N_X \langle \sigma v \rangle = \frac{\lambda_{aX} N_a N_X}{(1 + \delta_{aX})} \qquad (7)$$

& Energy generation is: $\epsilon = \dfrac{Qr}{\rho} = \dfrac{N_a N_X}{\rho} Q \dfrac{\langle \sigma v \rangle}{(1 + \delta_{aX})} \qquad (8)$

<u>Mean lifetime of a species:</u> $\left.\frac{\partial N_X}{\partial t}\right|_a = \frac{-N_X}{\tau_a(X)} = -r_{aX}$

or $\tau_a(X) = \frac{N_X}{(1+\delta_{aX})r_{aX}} = (\lambda_{aX} N_a)^{-1}$ & if species X is destroyed by several reactions, the total lifetime $\tau(X)$ is:

$$\tau(X) = \left[\sum \frac{1}{\tau_i(X)}\right]^{-1}$$

<u>Relative Velocity Distribution:</u>

Since, save in N★s & very late stages the ★ interior is non-degenerate, individual vel. dists. are Maxwellian:

$$N_1(v_1)\,d^3v_1 = N_1\left(\frac{m_1}{2\pi k T_1}\right)^{3/2}\exp\left(-\frac{m_1 v_1^2}{2kT_1}\right)d^3v_1$$

$\therefore r$ involves $\iint$ over:

$$N_1(v_1)\,d^3v_1\,N_2(v_2)\,d^3v_2 = N_1 N_2 \frac{(m_1 m_2)^{3/2}}{(2\pi k T)^3}\exp\left[\frac{-(m_1 v_1^2 + m_2 v_2^2)}{2kT}\right]d^3v_1\,d^3v_2$$

Convert this into reduced mass & relative velocities & eventually get:

$$N_1 N_2 \underbrace{\left\{\left(\frac{m_1+m_2}{2\pi kT}\right)^{3/2}\exp\left[\frac{-(m_1+m_2)V^2}{2kT}\right]d^3V\right\}}_{\textbf{CM vel.}}\underbrace{\left[\left(\frac{\mu}{2\pi kT}\right)^{3/2}\exp\left(\frac{-\mu v^2}{2kT}\right)d^3v\right]}_{\textbf{rel. vel.}}$$

$$\therefore r = \iint N_1(v_1)N_2(v_2)\,v\,\sigma(v)\,d^3v_1\,d^3v_2$$
$$= N_1 N_2 \int v\,\sigma(v)\left(\frac{\mu}{2\pi kT}\right)^{3/2}\exp\left(\frac{-\mu v^2}{2kT}\right)d^3v$$

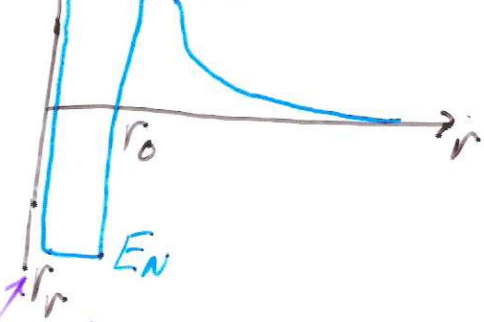
$$\Rightarrow \phi(v)\,dv = \left(\frac{\mu}{2\pi kT}\right)^{3/2}\exp\left(\frac{-\mu v^2}{2kT}\right)4\pi v^2\,dv \quad \text{or}$$

$$r_{12} = \frac{N_1 N_2 \langle\sigma v\rangle}{(1+\delta_{12})} = \frac{N_1 N_2\,4\pi}{(1+\delta_{12})}\left(\frac{\mu}{2\pi kT}\right)^{3/2}\int_0^\infty v^3\sigma(v)\exp\left(\frac{-\mu v^2}{2kT}\right)dv$$

and $\lambda = \langle\sigma v\rangle = 4\pi\left(\frac{\mu}{2\pi kT}\right)^{3/2}\int_0^\infty v^3\sigma(v)\exp\left(\frac{-\mu v^2}{2kT}\right)dv$

To do better, need expression for $\sigma(v)$. Will discuss later.

General Considerations



For $r > r_0$ Coulomb repulsion dominates: $E_C \sim \frac{e^2}{r_0} \simeq 1\,MeV$.
$r_r < r < r_0$ attractive nuclear potential dominates, but $\exists$ repulsive core.

For $T \sim 1.5 \times 10^7 K \Rightarrow \langle E \rangle \simeq \frac{3}{2} kT \sim 2\,keV \ll E_C \sim 1\,MeV$!

$\therefore$ Reactions must dominated by the tail of the thermal distribution w/ $E \gtrsim 10kT$ & assisted by QM tunneling — barrier penetration.

To estimate the minimum mass a ★ needs to ignite a particular type of fuel, note that:

$\rho \simeq \frac{M}{4R^3}$, $T_c \simeq \frac{1}{5} M G \mu m_H / kR$. At high ρ matter gets degenerate & $\therefore$ $kT < \epsilon_F = \frac{p_F^2}{2m_e} = \frac{\hbar^2 (3\pi^2 n_e)^{2/3}}{2m_e}$

since $p_F = \hbar (3\pi^2 n_e)^{1/3}$.

Since $n_e = n_p = \left(\frac{\rho}{m_H}\right)$ we need $\left(\frac{m_H}{\rho}\right)^{1/3} > \frac{\hbar (3\pi^2)^{1/3}}{(2m_e kT)^{1/2}}$

i.e. separation $> \lambda_e = \frac{\hbar}{p_e}$. Eliminate ρ & R in favor of M & T and finally get:

$$\mu^{3/2}\left(\frac{M}{M_\odot}\right) = \left[\frac{5^3 \hbar^3 (kT)^{3/2}}{4 m_H^2 m_e^{3/2} (\mu G)^3 M_\odot^2}\right]^{1/2} \simeq 2.9 \times 10^{-7}\, T^{3/4}$$

Knowing that:
$T_H \sim 8 \times 10^6 K \Rightarrow M \sim 0.1 M_\odot$ [really ~ 0.07]
$T_{He} \sim 10^8 K \Rightarrow M \sim 0.2 M_\odot$
$T_C \sim 4\text{-}7 \times 10^8 K \Rightarrow M \sim 0.5 M_\odot$
$T_O \sim 1.4\text{-}2 \times 10^9 K \Rightarrow M \sim 0.75 M_\odot$
$T_{Si} \sim 3\text{-}5 \times 10^9 K \Rightarrow M \sim 1.4 M_\odot$
} also inaccurate

Parameterization: $\epsilon = \epsilon_0 \rho^a T^n$ erg g^{-1}s^{-1}

$a \sim 1$ $\quad \underset{pp}{4} \lesssim n \lesssim \underset{3\alpha}{40}$ depending on particular reaction (n~18 for CNO)

p-p Chain

		τ^{*}(yr)	
(9)	$p + p \rightarrow d + e^+ + \nu_e$	7.9×10^9	$Q = 1.442$ MeV, but $\nu_e = .2$ & includes $e^+ + e^- = 1.022$

(Slowest & determines process rate) $^*(X = Y = 0.5,\ \rho = 100,\ T_6 = 15)$

(10)	$p + d \rightarrow {}^3He + \gamma$	4.4×10^{-8}	$Q = 5.493$ in γ

If $T \lesssim 10^7$K, this terminates here; but $T \gtrsim 10^7$K, then:

(11)	${}^3He + {}^3He \rightarrow {}^4He + 2p$	2.4×10^5	$Q = 12.859$

∴ Total release $= 2Q_9 + 2Q_{10} + Q_{11} = 26.729$ MeV/${}^4He$

But ∃ alternatives to (11), esp. as ${}^4He$ am't increases & $T > 2\times10^7$K:

(12)	${}^3He + {}^4He \rightarrow {}^7Be + \gamma$	9.7×10^5	$Q = 1.586$ ($\sim10^{-4}$ fraction)
(13a)	${}^7Be + e^- \rightarrow {}^7Li + \nu_e$	3.9×10^{-1}	$= 0.861$ (but $\langle\nu_e\rangle = .80$, 10^{-2} fraction)
(14a)	${}^7Li + p \rightarrow 2\,{}^4He$	1.8×10^{-5}	$= 17.347$
(13b)	${}^7Be + p \rightarrow {}^8B + \gamma$	6.6×10^1	$= 0.135$
(14b)	${}^8B \rightarrow {}^8Be^* + e^+ + \nu_e$	3×10^{-8}	$= 17.979$ w/ $\langle\nu_e\rangle = 7.2$
(14c)	${}^8Be \rightarrow 2\,He^4$		$= 0.095$

ppI : (9) (10) & (11)

ppII : (9) (10) (12) (13a) & (14a) increases as T↗, as age↗ & going from Pop II to Pop I

ppIII : (9) (10) (12) (13b) (14b) (14c)

Solar neutrino deficit: Homestake experiment (R. Davis)

depends on: ${}^{37}_{17}Cl + \nu_e \rightarrow {}^{37}_{18}Ar + e^-$ if $\nu_e \geq 5.1$ MeV

so only (14b) ν_e works (very rare, but rate $\sim \frac{1}{3}$ of prediction)

Other experiments: Galex, Sage have more sensitivity to ${}^7Be + e^- \rightarrow {}^7Li + \nu_e$ & even a little to (9)'s very dominant flux.

Either $T_{c\odot}$ < theory somehow (convective core, major pulsations, strong mag fields) or ν's have mass & flavors oscillate.

These nuclear reactions yield the following DEs for the am't of each species assuming no mixing, etc.

(15) $\frac{dH}{dt} = -2\lambda_{pp}\frac{H^2}{2} - \lambda_{pd}HD + 2\lambda_{33}\frac{({}^3He)^2}{2} - \lambda_{17}H\,{}^7Be - \lambda'_{17}H\,{}^7Li$

(16) $\frac{dD}{dt} = \lambda_{pp}\frac{H^2}{2} - \lambda_{pd}HD$ but since $\tau_D \ll 1\,yr$, rapid $\rightleftharpoons$ $\Rightarrow \frac{dD}{dt} = 0$ & $\lambda_{pd}HD = \lambda_{pp}\frac{H^2}{2}$

(17) $\frac{d\,{}^3He}{dt} = \lambda_{pd}HD - 2\lambda_{33}\frac{({}^3He)^2}{2} - \lambda_{34}\,{}^3He\,{}^4He$

(18) $\frac{d\,{}^4He}{dt} = \lambda_{33}\frac{({}^3He)^2}{2} - \lambda_{34}\,{}^3He\,{}^4He + 2\lambda_{17}H\,{}^7Be + 2\lambda'_{17}H\,{}^7Li$

(19) $\frac{d\,{}^7Be}{dt} = \lambda_{34}\,{}^3He\,{}^4He - \lambda_{e7}n_e\,{}^7Be - \lambda_{17}H\,{}^7Be$

(20) $\frac{d\,{}^7Li}{dt} = \lambda_{e7}n_e\,{}^7Be - \lambda'_{17}H\,{}^7Li$ (Since ${}^8B$ decays so quickly, it has been ignored)

${}^7Li$ & ${}^7Be$ also reach $\rightleftharpoons$ quickly (~ yrs) & then follow He growth $\Rightarrow$

(18) $\Rightarrow \frac{d\,{}^4He}{dt} \simeq \lambda_{33}\frac{({}^3He)^2}{2} + \lambda_{34}\,{}^3He\,{}^4He$

+ sign, despite consumption, since entire cycle returns two α's.

Abundance in $\rightleftharpoons$ of ${}^3He$ is always small & takes a long time to reach $\rightleftharpoons$. ∴ Approximately, (15) $\Rightarrow$

$$\frac{dH}{dt} = -3\lambda_{pp}\frac{H^2}{2} + 2\lambda_{33}\frac{({}^3He)^2}{2} - \lambda_{34}\,{}^3He\,{}^4He \quad \text{(using (18) \& (20))}$$

When ${}^3He$ does equilibrate, $\frac{d\,{}^3He}{dt} \simeq 0 \Rightarrow$

(21) $${}^3He_{,eq} = (2\lambda_{33})^{-1}\left\{-\lambda_{34}\,{}^4He + \sqrt{(\lambda_{34}\,{}^4He)^2 + 2\lambda_{pp}\lambda_{33}H^2}\right\}$$

Competition between PPI & PPII & PPIII is given by rate of α production by $\frac{PPI}{PPII + PPIII} = \frac{r_{33}}{r_{34}} = \frac{\lambda_{33}(^3He)^2/2}{\lambda_{34}\, ^3He\, ^4He} = \frac{1}{2}\frac{\lambda_{33}}{\lambda_{34}}\frac{^3He}{^4He}$

In $\rightleftarrows$ of ^3He: $$\frac{PPI}{PPII + PPIII} = \frac{-\lambda_{34} + \sqrt{\lambda_{34}^2 + 2\lambda_{pp}\lambda_{33}(H/^4He)^2}}{4\lambda_{34}}$$

You see that: (1) as ^4He or Y rises this branching ratio falls as $\frac{1}{Y}$ $\therefore$ Changes w/ t even if ρ & T stay the same.

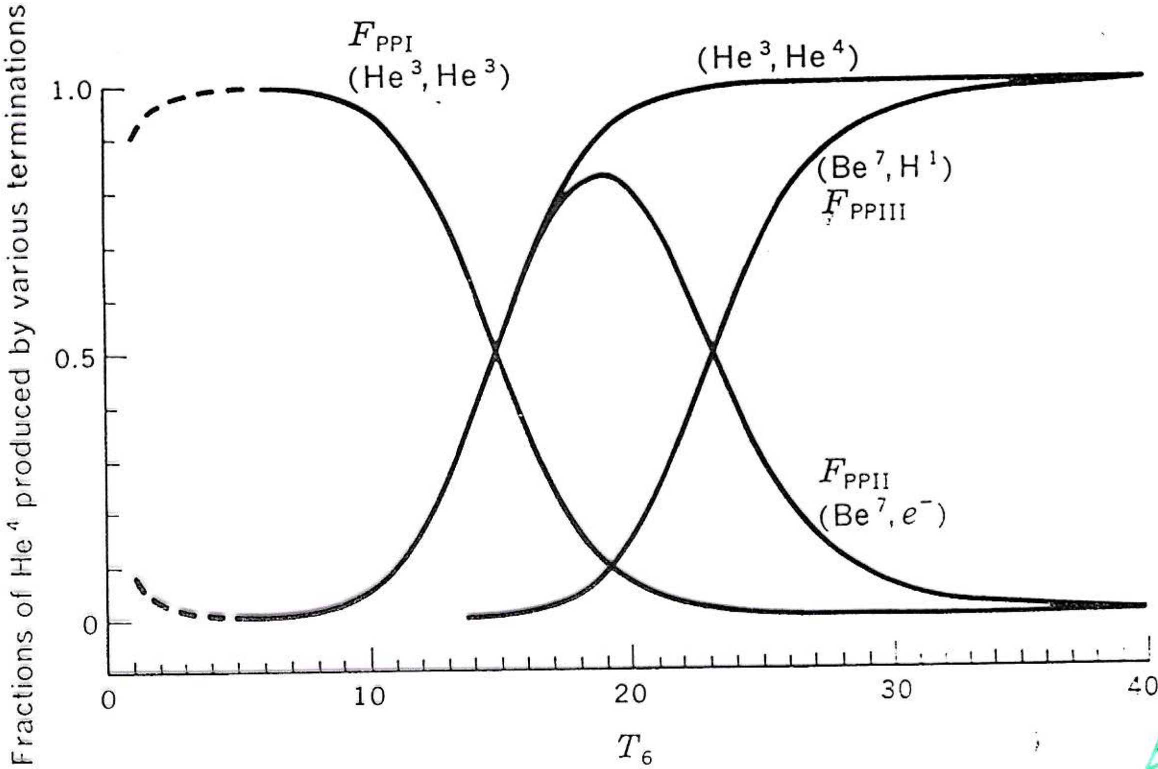
(2) It is also 0 at the beginning as $^3He = 0$ & rises to a maximum as $^3He \rightarrow {^3He}|_{equilibrium}$.

(3) It declines w/ $T \nearrow$ since $^3He/H|_{eq} \downarrow$ fast as $T \nearrow$.

The CNO bi-cycle is important in Pop I ☆s & less extreme Pop II as higher internal temperatures are reached (basically FS or earlier for Pop I). The bulk of the cycle uses ^{12}C as a catalyst for conversion of 4 ^1H into 1 ^4He, but a subsidiary part (bi-cycle) involves ^{16}O as the catalyst.

Clayton

54a



Eval for X = Y

Fig. 5-10 The fraction of the He^4 production due to PPI, PPII, and PPIII, respectively. The chains are assumed to be in equilibrium, and for the pur-

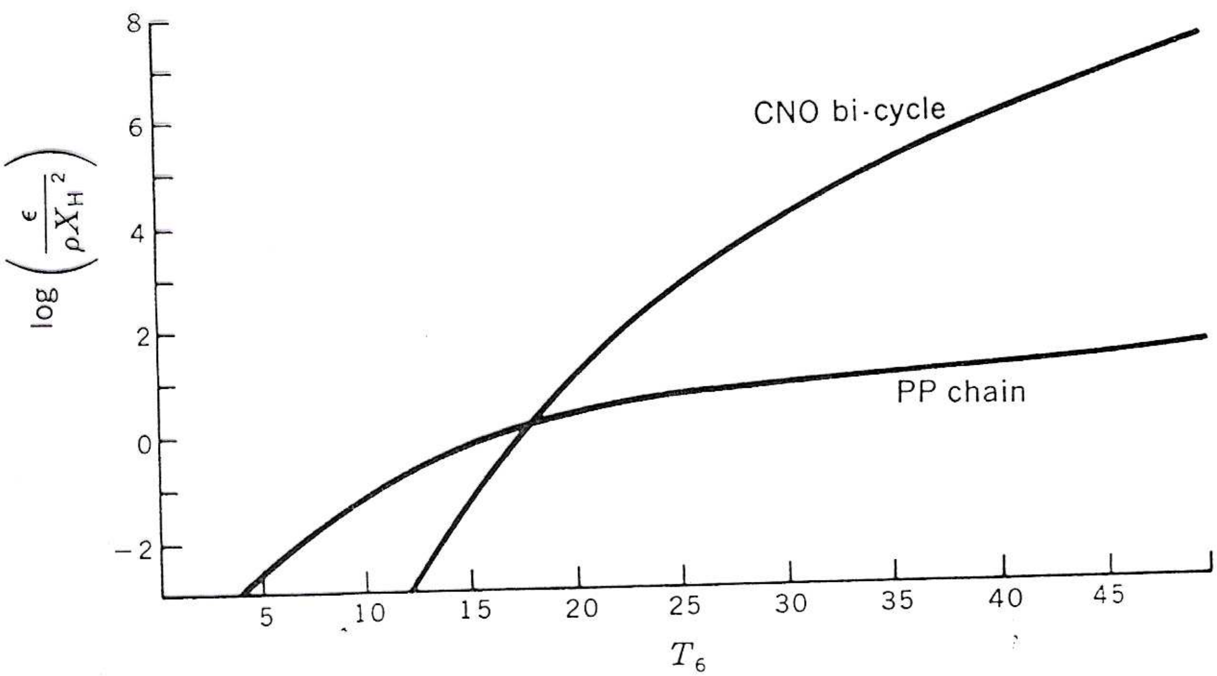


Fig. 5-16 A comparison of thermonuclear power from the PP chains and the CNO cycle. Both chains are assumed to be operating in equilibrium. The calculation was made for the choice $X_{CN}/X_H = 0.02$, which is representative of population I composition.