\[ \text{C} + \text{p} \rightarrow ^{12}\text{N} + \gamma \]
\[ ^{13}\text{N} \rightarrow ^{12}\text{C} + e^+ + 2\nu \]
\[ ^{13}\text{C} + \text{p} \rightarrow ^{14}\text{N} + \gamma \]
\[ ^{14}\text{N} + \text{p} \rightarrow ^{15}\text{O} + \gamma \]
\[ ^{15}\text{O} \rightarrow ^{15}\text{N} + e^+ + \nu_e \]
\[ ^{15}\text{N} + \text{p} \rightarrow ^{12}\text{C} + ^{4}\text{He} \]

\[ \text{CN} \text{ cycle} \quad (\tau = 870 \text{ s}) \]

\[ \text{But 0.04}\% \text{ branching ratio to:} \]
\[ ^{15}\text{N} + \text{p} \rightarrow ^{16}\text{O} + \gamma \]
\[ ^{16}\text{O} + \text{p} \rightarrow ^{17}\text{F} + \gamma \]
\[ ^{17}\text{F} \rightarrow ^{17}\text{O} + e^+ + \nu_e \]
\[ ^{17}\text{O} + \text{p} \rightarrow ^{14}\text{N} + ^{4}\text{He} \]

\[ \text{2nd cycle (ON)} \]

99.96\% of the time

The seed nuclei are believed to mainly be \(^{12}\text{C}\) & \(^{16}\text{O}\) since they are the main products of Helium fusion.

Analysis of the change & rates indicate \(^{12}\text{C}\) & \(^{16}\text{O}\) also reaches \(\approx\) (after longer time) where most is \(^{14}\text{N}\).

Key results in \text{CNO} bi-cycle (through rate equals similar to those dealt with already for \text{PP} chains):

1. For short times, while \text{CN} cycle \(\approx\) the \(^{16}\text{O}\) can be ignored since very little \(^{16}\text{O}\) will burn in that time (excepting \(^{17}\text{O}\), which is very rare anyway).

2. Times long enough to include \(^{16}\text{O}\) burning & transfer of \text{CN} nuclei to the \text{ON} cycle through the rare branch occur after the \text{CN} nuclei have already reached \(\approx\).
Reaction rates

Coulomb barrier: \( \frac{Z_1 Z_2 e^2}{r^*_e} \), \( r_e^* = \text{radius of combined nucleus} \)

But for \( T = 10^7 \text{K} \), \( kT/8\pi^2 = 10^{-3} \) of barrier height.

Barrier penetration turns out to depend on faster get closer:

\[ (22) \quad \exp \left( -\frac{2\pi Z_1 Z_2 e^2}{mv} \right) \quad \text{so } \sigma \text{ must have a similar dependence.} \]

We can motivate (22) as follows: chance of tunneling = \( \exp \left( -\frac{E_c}{E_{KE}} \right) = \exp \left( -\frac{Z_1 Z_2 e^2}{mv^2} \right) \) but \( r_0 = \frac{\hbar}{m v} \) is like the de Broglie wavelength. \( \hbar \)

and we get \( E_c = \frac{Z_1 Z_2 e^2}{mv} \), only off by \( \pi r \) from (22).

However, the prob. of interaction also depends on a "QM area": \( (23) \quad \pi r^2 = \pi \left( \frac{\hbar}{m v} \right)^2 \propto \frac{1}{E_{KE}} \) (if NR)

At low values of \( E_{KE} \) both (22) and (23) vary rapidly.

and the rest of the cross-section can be taken to vary more slowly.

For non-resonant reactions:

\[ (24) \quad \sigma(E) = \frac{S(E)}{E} \exp \left( -\frac{b}{\sqrt{E}} \right) \quad \text{where} \]

\( b = 31.28 Z_1 Z_2 A^*(\text{keV})^{1/2} \) with \( A = \frac{A_1 A_2}{A_1 + A_2} \), reduced atomic mass

since: \( \nu(E) = \frac{\sqrt{2E}}{\pi} \)

We want \( \langle N \rangle = \int_0^\infty \sigma(v) \phi(v) dv = \int_0^\infty V(E) \sigma(E) f(E) dE \quad (25) \)

with \( f(E) = \frac{2}{\sqrt{\pi}} \frac{E}{kT} e^{-E/kT} dE \) \( \text{The Maxwell distr in E space} \)

\[ (26) \langle N \rangle = \left( \frac{2}{\sqrt{\pi}} \right)^{1/2} (kT)^{-3/2} \int_0^\infty S(E) \exp \left( -\frac{E - \frac{1}{2} b}{kT} \right) dE \]

Use the method of steepest descent to perform the integral.
Gamow peak is a Gaussian & $S(E) \propto S(E_0)$. Then \( E_0 = \left( \frac{\hbar k T}{2} \right)^{2/3} \) and \( \Delta = \frac{4}{\sqrt{3}} \left( \frac{E_0 k T}{\hbar^2} \right)^{1/2} = \frac{4}{\sqrt{3}} \left( \frac{1}{2} \right)^{1/3} (kT)^{5/6} \) max
\[
\text{we get } \langle \sigma v \rangle = \frac{8}{\pi \mu m} \left( \frac{kT}{2} \right)^{3/2} \exp \left( -\frac{3E_0}{kT} \right) \Delta \frac{\sqrt{\pi}}{2}
\]
(27)
\[E_0 \text{ comes from } \frac{dE}{dt} = \frac{1}{\hbar^2} \left( \frac{E_0}{kT} \right)^{3/2} \]
To find max of joint curve!

Let:
\[n = \frac{3E_0}{kT} = 3 \left( \frac{kT}{2} \right)^{2/3} (kT)^{-1/3} = BT^{-1/3}
\]
(28), so
\[\tau_{12} = \langle \sigma v \rangle = \frac{8E}{\sqrt{3}} \frac{S(E_0)}{m b} \eta^2 e^{-\eta^2}
\]
the reaction rate per pair
\[= \frac{7.20 \times 10^{-19}}{A_{14} Z_i^2} S_0 (\text{keV barns}) \eta^2 e^{-\eta^2}
\]
(1 barn = $10^{-24}$ cm$^2$

and
\[\tau_{12} = \frac{N_i N_2 \tau_{12}}{(1 + \delta_{12})}
\]
\[\Delta T \text{ is usually fairly small, so that } T = T_0 \left( 1 + \frac{\eta^2}{3} \right)
\]
\[\frac{n}{n_0} = \left( \frac{T}{T_0} \right)^{-1/3} \approx 1 - \frac{1}{3} \eta
\]
\[\frac{\tau}{\tau_0} = \frac{S(E_0)}{S(E_0)}_p \left( \frac{m}{m_0} \right)^2 e^{-\left( \eta - n_0 \right)} = \left( \frac{m}{m_0} \right)^2 e^{-\eta_0} \left( e^{\eta_0 - 1} \right) \approx (1 - \frac{2}{3} \eta)(1 + \frac{2}{3} \eta)
\]
\[\eta = \frac{2 m - 2}{3}
\]

If \( \frac{\tau}{\tau_0} \propto \left( \frac{T}{T_0} \right)^n \) we conclude \( n = \frac{2 - 2}{3} \) or \( \tau = \tau_0 \left( \frac{T}{T_0} \right)^n \)

and
\[\tau = \frac{E \rho T^n}{n} = \frac{N_i N_2 \rho T^n}{(1 + \delta_{12})}
\]

In pp chain \( T_0 = 10^7 K, n = 4.5 \),

CN cycle \( T_0 = 2.5 \times 10^7 K, n = 16.7 \)
\[3 \alpha \rightarrow E_3 \alpha = E_0 \rho T^{30} \]
\[3 \alpha \rightarrow [3 \alpha \rightarrow ^6 \text{He} \rightarrow ^{12} \text{C} + \alpha]
\]
\[^{12} \text{C} + ^{12} \text{C} \rightarrow ^{24} \text{Mg} + \alpha, ^{23} \text{Na} + \rho, ^{23} \text{Mg} + n, ^{20} \text{Ne} + \alpha
\]
\[E_{cc} = E_0 \rho T^{32}
\]
Resonant Reactions

Big problem: nuclear reaction rates are usually measured @ energies $\gg kT$, and must be extrapolated down to those $T$'s => real uncertainties. Also, most reactions proceed with excited compound nuclei forming & then breaking up in a smooth fashion. But some reactions are resonant & proceed through specific energy levels w/ very high probability of capture. Even if $E \neq E_r$ it wings to resonance that could dominate $S(E)$

\[
\begin{align*}
S(E) &= \frac{657}{A} \frac{\omega \Gamma_1(E) \Gamma_2}{(E-E_r)^2 + (\Gamma_2/\sqrt{2})^2} \exp\left(31.28 Z_1 Z_2 A^{1/2} E^{-1/2}\right) \text{ keV barns (31)}
\end{align*}
\]

In the above case, $^{12}\text{C}(p, x)^{13}\text{N}$: $\Gamma_1 = \Gamma_p$ and $\Gamma_2 = \Gamma_n = \text{const}$

This comes from the cross-section depending on $E$ through penetration factor for single-level, being ($l$ = ang. mom. state)

\[
\begin{align*}
\Gamma_{l, l'}(a, b) &= (2l+1) \pi \frac{1}{2} \omega \frac{\Gamma_a \Gamma_b}{(E-E_r)^2 + (\Gamma_2/\sqrt{2})^2} \begin{cases} 
\text{Compton} & (l+\frac{1}{2}) \text{ ang. mom. of the resonance} \\
\text{Compton} & (l+\frac{1}{2}) \text{ ang. mom. of the resonance} \\
\end{cases}
\end{align*}
\]

with $\omega = \frac{2l+1}{(2l+1)(2l_2+1)}$
Electron Screening: At high enough $p$ then reactions can be forced: electrons screen nuclei $\Rightarrow$ lower effective $Z$'s $\Rightarrow$ lower Coulomb barrier. Also, if very high $p$ then uncertainty principle from degeneracy $\Rightarrow$ 

$$E \sim \frac{\hbar^2}{2ma^2} = \frac{\hbar^2}{2m_H} \rho^{2/3}$$

and this zero-pt energy could initiate pycnonuclear reactions for $\rho > 5 \times 10^{14} \text{ g cm}^{-3}$ for H, $\rho > 8 \times 10^{8} \text{ for He}$, and $\rho > 6 \times 10^{8} \text{ g cm}^{-3}$ for $^{12}\text{C}$ - rare, but conceivable in accretion disks & WD interiors. What is the screened potential?

Approx answer is: Z.e immersed in Ne initially uniform.

$$\nabla^2 \phi_s (r) = \frac{4\pi e}{r} \left[ N_e(r) - N_e \right]$$

screened distribution

Chemical potential of an e @ high $T$, $\mu \gg kT$:

$$\frac{\mu_e}{kT} = \ln \left[ \frac{N_e}{Z} \left( \frac{2\pi \hbar^2}{m_e k T} \right)^{3/2} \right]$$

but in $\phi_s$ it becomes:

$$\mu_e = \mu_0 + e\phi_s (r) \quad \text{or} \quad \mu_e - kT = \ln \left[ \frac{N_e(r)}{Z} \left( \frac{2\pi \hbar^2}{m_e k T} \right)^{3/2} \right]$$

$$e\phi_s (r) = \mu_e - \mu_0 = \ln \left( \frac{N_e(r)}{2} \right)^{3/2} - \ln \left( \frac{N_e}{2} \right)^{3/2} \ln \left( \frac{N_e(r)}{N_e} \right)$$

or

$$\exp \left( \frac{e\phi_s}{kT} \right) = \frac{N_e(r)}{N_e}$$

If $e\phi_s / kT \ll 1$,

$$1 + \frac{e\phi_s}{kT} \approx \frac{N_e(r) + N_e(r) - N_e}{N_e} \Rightarrow N_e(r) - N_e \approx N_e e\phi_s (r)$$

\text{Into (32)} $\Rightarrow$ 

$$\nabla^2 \phi_s (r) = \left( \frac{4\pi e N_e e^2}{kT} \right) \phi_s (r)$$
or \[ \nabla^2 \phi_s(r) = \frac{Z_1 Z_2 e}{r} \phi_s(r) \]

the Debye length

This has the solution: \[ \phi_s(r) = \frac{Z_1 e}{r} \exp \left( -\frac{r}{\lambda D} \right) \]

is the screened potential. If \( \lambda D \) is small the screening is strong, if \( \lambda D \) is large, the screening is weak.

If \( r_0 \ll \lambda D \), then \[ \phi_s(r) = \phi(r) - Z_1 Z_2 \left( \frac{4\pi n e^2}{kT} \right)^{1/2} \]

\[ \Rightarrow \text{The screened potential energy acting on an ion with } Z_2 e \text{ near the nucleus is: } U_s(0) = -Z_1 Z_2 e^2 \left( \frac{4\pi n e^2}{kT} \right)^{1/2} \]

To see effects on cross-sections, recall:

\[ \langle \sigma v \rangle = \int_0^\infty \sigma(v) v f(v) dv \]

and note that w/o screening the ion's motion is produced by:

\[ E - V(r) = E - \frac{Z_1 Z_2 e^2}{r} \]

But w/ screening, \[ U_s(r) = V(r) + U_s(r) \] so motion comes from:

\[ E - U_s(r) = E - U_s(r) - V(r) \Rightarrow \text{Replace } E \text{ by } E - U_s(r) \]

\[ \Rightarrow \langle \sigma v \rangle_s = \int_0^\infty \sigma [E - U_s(r)] v f(v) dv \] (35)

If weak screening, then, since pot. varies slowly w/ in the screening cloud: \[ U_s(r) \approx U_s(0) \] (36)

w/ \[ r_0 = \frac{Z_1 Z_2 e^2}{E_0} \] as the distance of closest approach.

Use the fact that \( E_1, E_0 \gg U_s(0) \Rightarrow \) use (36) in (35)

\[ \langle \sigma v \rangle_s = \int_0^\infty \sigma [E - U_s(0)] v f(v) dv = \langle \sigma v \rangle \exp \left[ -\frac{U_s(0)}{kT} \right] \]

Plug (34) into (37) to get:

\[ -U_s(0) = \frac{Z_1 Z_2 e^2}{kT} \left( \frac{4\pi n e^2}{kT} \right)^{1/2} \]

\[ = 0.188 Z_1 Z_2 \sqrt{\frac{12}{N_c} \frac{1}{T}} \]

where \( f = \frac{Z_1}{2} + \frac{Z_2}{2} \), \( Z_1 \ll Z_2 \) (of \( T \approx 0.05 \))

so, as long as \( N_c \) is small: \[ \exp \left( \frac{U_s(0)}{kT} \right) \approx 1 + 0.188 Z_1 Z_2 \sqrt{\frac{12}{N_c} \frac{1}{T}} \]

(39)

This correction is 10% for PP but up to 50% for CNO (Z_2 much higher), so must include it.
Based on: 1) $E = E_0 \rho^{a} T^{n}$
2) $K = K_0 \rho^{b} T^{-3.5}$
3) $P \propto \rho T$ or $P \propto T^4$
4) No convection!

Then, if $(P, T, \rho, L)$ are $f(u,v)$ for a given $M$ & $\mu$

...simple results of scaling to a new $M'$:

\[
\frac{d\rho}{dr} = -\frac{6\mu P}{r^2} \quad \frac{dm}{dr} = 4\pi r^2 \rho
\]

\[
\frac{dT}{dr} = -\frac{3K_0 \rho^2 T^{-3.5}}{16\pi \rho c r^2 T^3} l \quad \frac{dR}{dr} = 4\pi r^2 \rho^2 e_0 \rho T^{-n}
\]

$P = \frac{\rho k T}{\mu m H}$ or $P = \frac{1}{3} \frac{a T^4}{\mu m H}$

Basically the dimensional analysis we performed earlier leads to:

\[
\rho_2(x) = \left(\frac{M_2}{M_1}\right)^3 \rho_1(x) \quad \text{with} \quad x = \frac{R_1}{R_2} = \frac{r_1}{r_2}
\]

\[
P_2(x) = \left(\frac{M_2}{M_1}\right)^2 \left(\frac{R_1}{R_2}\right)^4 P_1(x)
\]

\[
T_2(x) = \frac{M_2}{M_1} \frac{R_1}{R_2} T_1(x) \quad \text{if gas pressure dominated.}
\]

& homologous, only if $\frac{M_2}{M_1} \neq f(u)$ a full analog requires:

\[
L_2(x) = \frac{E_{02}}{E_{01}} \left(\frac{M_2}{M_1}\right)^{n+2} \left(\frac{R_1}{R_2}\right)^{n+3} L_1(x)
\]

So $\frac{E_{02}}{E_{01}}$ also must be const. in $x$. Demanding consistency for $\frac{dT}{dr} \Rightarrow$ \[
\frac{E_{02}}{E_{01}} \frac{K_02}{K_01} \left(\frac{M_2}{M_1}\right)^{n+2.5} \left(\frac{R_1}{R_2}\right)^{n+2.5} = 1
\]

If $n = 5$ is assumed typical of pp chain (LMS) & $n = 18$ for CNO (VMS) & with $P_9$ for LMS & $P_{rad}$ for VMS [i.e., extreme VMS] then:
Lower Main Sequence
\[ R \sim \left( \frac{\text{K}_o \text{E}_o}{\text{G}_u} \right)^{2/15} \left( \frac{\text{G}_u}{\text{M}} \right)^{-11/15} \text{M}^{11/15} \]
\[ T \sim \left( \frac{\text{K}_o \text{E}_o}{\text{G}_u} \right)^{2/15} \left( \frac{\text{G}_u}{\text{M}} \right)^{4/13} \text{M}^{12/15} \]
\[ L \sim \text{K}_o^{-1/6} \text{E}_o^{5/6} \left( \frac{\text{G}_u}{\text{M}} \right)^{-4/13} T^{1/2} \text{M}^{0.81/12} \]
\[ \text{L} \sim \text{K}_o^{-1/6} \text{E}_o^{5/6} \left( \frac{\text{G}_u}{\text{M}} \right)^{23/12} \text{M}^{27/15} \text{or} \text{L} \sim \text{M}^{5.4} \]

Upper Main Sequence
\[ R \sim \left( \text{K}_o \text{E}_o \right)^{3/4} \text{G} \left( \frac{2}{3} \right)^{164} \text{M}^{37/82} \]
\[ T \sim \left( \text{K}_o \text{E}_o \right)^{-2/3} \text{G} \left( \frac{2}{3} \right)^{164} \text{M}^{2/41} \]
\[ L \sim \text{K}_o^{-1/2} \text{E}_o^{3/2} \text{G}^{-2} T^{125/14} \text{L} \sim \text{LHS} \text{of H-R diagram}! \]
\[ \text{L} \sim \text{K}_o^{-1/2} \text{E}_o^{3/2} \text{G}^{-2} T^{125/14} \text{L} \sim \text{LMS obs. L} \sim T^{23/14} \text{near to the LMS obs. L} \sim T^{7} \]
\[ \therefore \text{H-R diagram is not broadened too much by composition changes.} \]

Real LMM is inbetween \( \Rightarrow \) GP more important than RP but type of reaction also important.

These kinds of considerations yield a strong limit on G varying with time to \( \frac{\Delta G}{G} \leq 10^{-11} \text{yr}^{-1} \)
on LMS dominance of composition through L\( \text{yu}_{23/3} \)
\[ T \sim \text{yu}_{4/3} \Rightarrow L \sim T^{23/14} \text{near to the LMS obs. L} \sim T^{7} \]
For a more general treatment, see T.R. Carson, Observatory Vol 106, p 71-72 (1986)