$$^{12}C + p \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^+ + \nu_e \quad (\tau = 870s)$$
$$^{13}C + p \rightarrow {}^{14}N + \gamma$$
$$^{14}N + p \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^+ + \nu_e \quad (\tau = 178s)$$
$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He$$

CN cycle

99.96% of the time

But 0.04% branching ratio to:

$$^{15}N + p \rightarrow {}^{16}O + \gamma$$
$$^{16}O + p \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^+ + \nu \quad (\tau = 95s)$$
$$^{17}O + p \rightarrow {}^{14}N + {}^{4}He$$

2nd cycle (ON)

The seed nuclei are believed to mainly be ^{12}C & ^{16}O since they are the main products of Helium fusion. Analysis of the changes & rates indicate C&N → 98% ^{14}N in equilibrium & N&O also reaches ⇌ (after longer time) where most is ^{14}N.

Key results in CNO bi-cycle (through rate eqns similar to those dealt with already for PP chains):

(1) For short times, while CN cycle → ⇌ the O can be ignored since very little O will burn in that time (excepting ^{17}O, which is very rare anyway).

(2) Times long enough to include O burning & transfer of CN nuclei to the ON cycle through the rare branch, occur after the CN nuclei have already reached ≈ ⇌.

Reaction Rates

Coulomb barrier: $\dfrac{Z_1 Z_2 e^2}{r_*}$, r_* = radius of combined nucleus

But for $T = 10^7 K$, $kT = 861\,eV \simeq 10^{-3}$ of barrier height.

Barrier penetration turns out to depend on (faster get closer):

(22) $\exp\left(-\dfrac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$ so σ must have a similar dependence.

We can motivate (22) as follows: chance of tunneling ~

$\exp\left(-\dfrac{E_c}{E_{KE}}\right) = \exp\left(-\dfrac{\frac{Z_1 Z_2 e^2}{r_0}}{\frac{mv^2}{2}}\right)$ But $r_0 \simeq \dfrac{\hbar}{p} = \dfrac{\hbar}{mv}$ is like the deBroglie wavelength

& we get $\dfrac{E_c}{E_{KE}} \simeq \dfrac{2 Z_1 Z_2 e^2}{\hbar v}$, only off by π from (22).

However, the prob. of interaction also depends on a "QM area": (23) $\pi \lambda^2 = \pi \left(\dfrac{h}{p}\right)^2 \propto \dfrac{1}{E_{(KE)}}$ (if NR)

At low values of $E_{(KE)}$ both (22) & (23) vary rapidly & the rest of the cross-section can be taken to vary more slowly: For non-resonant reactions:

(24) $\sigma(E) \equiv \dfrac{S(E)}{E} \exp\left(-b/E^{1/2}\right)$ **where**

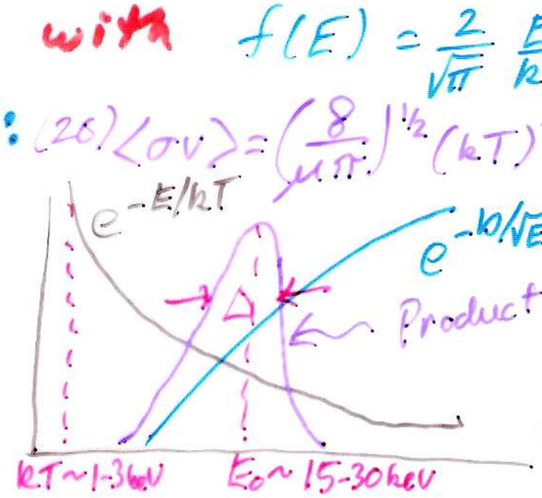
$b = 31.28\, Z_1 Z_2 A^{1/2} (keV)^{1/2}$ **with** $A = \dfrac{A_1 A_2}{A_1 + A_2} = \dfrac{\mu}{m_u}$ **reduced atomic mass**

since: $v(E) = \sqrt{\dfrac{2E}{\mu}}$

We want $\langle \sigma v \rangle = \int_0^\infty v\,\sigma(v)\,\phi(v)\,dv = \int_0^\infty v(E)\,\sigma(E)\,f(E)\,dE$ (25)

with $f(E) = \dfrac{2}{\sqrt{\pi}} \dfrac{E}{kT} e^{-E/kT} \dfrac{dE}{(kTE)^{1/2}}$ **the Maxwell dist in E space**

$\therefore$ (26) $\langle \sigma v \rangle = \left(\dfrac{8}{\mu \pi}\right)^{1/2} (kT)^{-3/2} \int_0^\infty S(E) \exp\left(-\dfrac{E}{kT} - \dfrac{b}{\sqrt{E}}\right) dE$



Use the method of steepest descent to perform the integral

Gamov peak is ~ Gaussian & $S(E) \simeq S(E_0)$. Then w/

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \text{ and } \Delta = \frac{4}{\sqrt{3}}(E_0 kT)^{1/2} = \frac{4}{\sqrt{3}}\left(\frac{b}{2}\right)^{1/3}(kT)^{5/6}$$

max ~ FW @ 1/e

we get $\langle \sigma v \rangle \simeq \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \exp\left(-\frac{3E_0}{kT}\right) \Delta \frac{\sqrt{\pi}}{2}$ (27)

[E_0 comes from $\frac{d}{dE}\left(\frac{E}{kT} + \frac{b}{E^{1/2}}\right) = 0 = \frac{1}{kT} - \frac{1}{2} b E_0^{-3/2}$] To find max of joint curve!

Let: $\eta = \frac{3E_0}{kT} = 3\left(\frac{b}{2}\right)^{2/3}(kT)^{-1/3} = BT^{-1/3}$ (28), so

$$\lambda_{12} = \langle \sigma v \rangle = \frac{8\sqrt{2}}{9\sqrt{3}} \frac{S(E_0)}{mb} \eta^2 e^{-\eta} \quad (29)$$

the reaction rate per pair

$$= \frac{7.20 \times 10^{-19}}{A^{1/2} Z_1 Z_2} S_0(\text{keV barns}) \, \eta^2 e^{-\eta}$$

1 barn $= 10^{-24}$ cm^2

and $r_{12} = \frac{N_1 N_2 \lambda_{12}}{(1+\delta_{12})}$

$\frac{\Delta T}{T}$ is usually fairly small, so that! $T = T_0(1+\xi)$

$\therefore \frac{\eta}{\eta_0} = \left(\frac{T}{T_0}\right)^{-1/3} \simeq 1 - \frac{1}{3}\xi$ so

$$\frac{r}{r_0} = \frac{S(E_0)_T}{S(E_0)_{T_0}} \left(\frac{\eta}{\eta_0}\right)^2 e^{-(\eta-\eta_0)} \simeq \left(\frac{\eta}{\eta_0}\right)^2 e^{-\eta_0(\eta/\eta_0 - 1)} \simeq \left(1 - \frac{2}{3}\xi\right)\left(1 + \eta_0 \frac{\xi}{3}\right)$$

$$\simeq 1 + \xi\left(\frac{\eta_0 - 2}{3}\right)$$

If $\frac{r}{r_0} \propto \left(\frac{T}{T_0}\right)^n$ we conclude $n = \left(\frac{\eta_0 - 2}{3}\right)$ or $\lambda = \lambda_0\left(\frac{T}{T_0}\right)^n$

and $\epsilon = \epsilon_0 \rho T^n = N_a N_x Q \frac{\lambda_0}{T_0^n} T^n$ (30)

In pp chain $T_0 \simeq 10^7$ K, $n \simeq 4.5$

CNO cycle $T_0 = 2.5 \times 10^7$ K, $n \simeq 16.7$

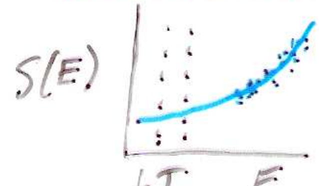
3-α : $\epsilon_{3\alpha} = \epsilon_0 \rho^2 T^{30}$ [$3\,{}^4He \to {}^{12}C + \gamma$]

${}^{12}C + {}^{12}C \to {}^{24}Mg + \gamma$, ${}^{23}Na + p$, ${}^{23}Mg + n$, ${}^{20}Ne + \alpha$

$$\epsilon_{CC} = \epsilon_0 \rho T^{32}$$

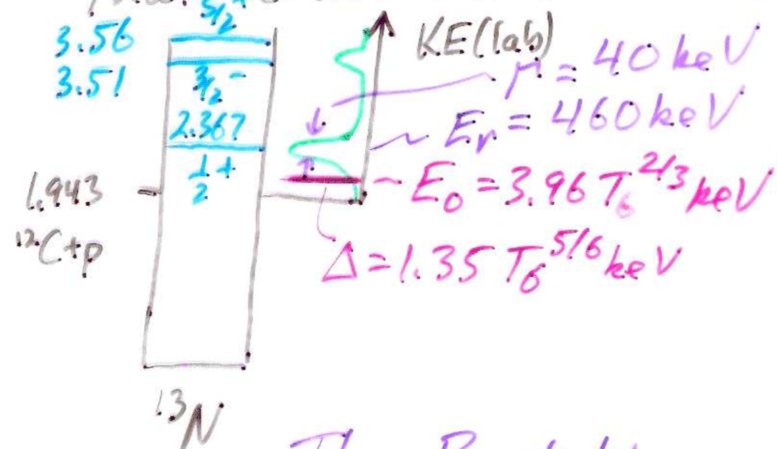
Resonant Reactions

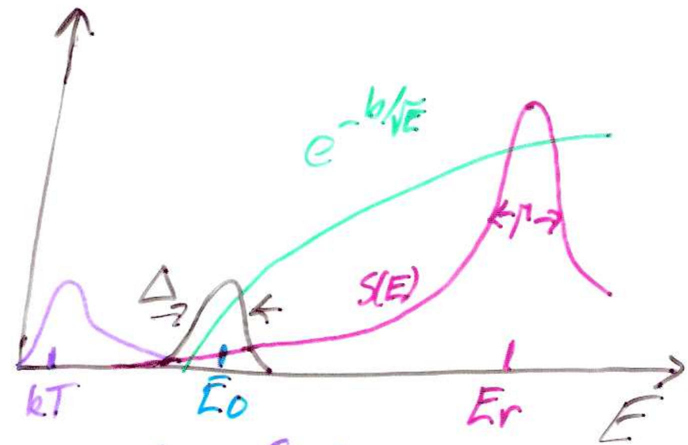
Big problem: nuclear reaction rates are usually measured @ energies $\gg kT_\star$ & must be extrapolated down to those T's $\Rightarrow$ real uncertainties.



Also, most reactions procede with excited compound nuclei forming & then breaking up in a smooth fashion.

But some reactions are resonant & procede through specific energy levels w/ very high probability of capture. Even if $E \neq E_r$ $\exists$ wings to resonance that could dominate S(E)





The Breit-Wigner cross-section factor is:

$$S(E) = \frac{657}{A} \frac{\omega \Gamma_1(E)\Gamma_2}{(E-E_r)^2+(\Gamma/2)^2} \exp\left(31.28 Z_1 Z_2 A^{1/2} E^{-1/2}\right) \text{ keV barns} \quad (31)$$

In the above case, $^{12}C(p,\gamma)^{13}N$: $\Gamma_1 = \Gamma_p$ and $\Gamma_2 = \Gamma_\gamma \simeq$ const

Γ_p depends on E through penetration factor

This comes from the cross-section for single-level, being (l = ang. mom. state)

$$\sigma_{r,l}(a,b) = (2l+1)\pi \lambda\!\!\!^{-2} \omega \frac{\Gamma_a \Gamma_b}{(E-E_r)^2+(\Gamma/2)^2}$$

$\lambda\!\!\!^{-}$: Compton λ

with $\omega = \dfrac{2J+1}{(2J_1+1)(2J_2+1)}$ — J ang. mom. of the resonance & J_1, J_2 of particles 1 & 2

Electron Screening: At high enough ρ then reactions can be forced: electrons sceen nuclei $\Rightarrow$ lower effective Z's $\Rightarrow$ lower Coulomb barrier. Also, if very high ρ then uncertainty principle from degeneracy $\Rightarrow E_0 \simeq \frac{\Delta p^2}{2m}$ & $\Delta p \gtrsim \frac{\hbar}{\Delta x}$ ~ spacing, a

$\Rightarrow E_0 \simeq \frac{\hbar^2}{2ma^2} \simeq \frac{\hbar^2}{2m_H^{5/3}} \rho^{2/3}$ and this zero-pt energy could initiate pycnonuclear reactions for $\rho > 5\times10^{14}$ g cm^{-3} for H, $\rho > 8\times10^{8}$ for He and $\rho > 6\times10^{9}$ g cm^{-3} for ^{12}C – rare, but conceivable in accretion disks & WD interiors.

What is the screened potential?

Approx. answer is: $Z_1 e$ immersed in n_e initially uniform.

$$\nabla^2 \phi_s(r) = 4\pi e\left[n_e(r) - n_e\right]$$

$n_e(r)$ ~ screened distribution

Chemical potential of an e @ high T, $\mu \Rightarrow kT \gg \mu$:

$$\frac{\mu_0}{kT} = \ln\left[\frac{n_e}{2}\left(\frac{2\pi\hbar^2}{m_e kT}\right)^{3/2}\right] \quad \text{but in } \phi_s \text{ it becomes:}$$

$$\mu_e = \mu_0 + e\phi_s(r) \quad \text{or} \quad \frac{\mu_e}{kT} = \ln\left[\frac{n_e(r)}{2}\left(\frac{2\pi\hbar^2}{m_e kT}\right)^{3/2}\right]$$

$$\therefore \frac{e\phi_s(r)}{kT} = \frac{\mu_e}{kT} - \frac{\mu_0}{kT} = \ln\left(\frac{n_e(r)}{2}\left(\;\right)^{3/2}\right) - \ln\left(\frac{n_e}{2}\left(\;\right)^{3/2}\right) = \ln\left(\frac{n_e(r)}{n_e}\right)$$

or $$\exp\left(\frac{e\phi_s}{kT}\right) = \frac{n_e(r)}{n_e} \qquad \text{If } \frac{e\phi_s}{kT} \ll 1,$$

$$1 + \frac{e\phi_s}{kT} \simeq \frac{n_e + n_e(r) - n_e}{n_e} \Rightarrow n_e(r) - n_e \simeq n_e \frac{e\phi_s(r)}{kT}$$

$$\therefore \text{ Into (32)} \Rightarrow \nabla^2 \phi_s(r) = \left(\frac{4\pi n_e e^2}{kT}\right)\phi_s(r)$$

or $\nabla^2 \phi_s(r) = \lambda_D^{-2} \phi_s(r)$ (33)

$\sim$ the Debye length

This has the solution: $\phi_s(r) = \frac{Z_1 e}{r} \exp{-r/\lambda_D}$ is the screened potential. If λ_D is small the screening is strong, if λ_D is large, the screening is weak.

If $r_0 \ll \lambda_D$, then $\phi_s(r) \simeq \phi(r) - Z_1 e \left(\frac{4\pi n_e e^2}{kT}\right)^{1/2}$

$\therefore$ The screened potential energy acting on an ion w/ $Z_2 e$ near the nucleus is: $U_s(0) = -Z_1 Z_2 e^2 \left(\frac{4\pi e^2 n_e}{kT}\right)^{1/2}$ (34)

To see effects on cross-sections, recall:

$\langle \sigma v \rangle = \int_0^\infty \sigma(E) v(E) f(v) dv$ and note that w/o screening the ion's motion is produced by:

$E - V(r) = E - \frac{Z_1 Z_2 e^2}{r}$ But w/ screening,

$V_s(r) = V(r) + U_s(r)$ so motion comes from:

$E - V_s(r) = E - U_s(r) - V(r) \Rightarrow$ Replace E by $E - U_s(r)$

$\therefore \langle \sigma v \rangle_s = \int_0^\infty \sigma[E - U_s(r)] V[E - U_s(r)] f(v) dv$ (35)

If weak screening, then, since pot. varies slowly w/in the screening cloud: $U_s(r) \simeq U_s(0)$ (36)

w/ $r_0 = \frac{Z_1 Z_2 e^2}{E_0}$ as the distance of closest approach.

Use the fact that $E_1, E_0 \gg U_s(0) \Rightarrow$ use (36) in (35)

$\langle \sigma v \rangle_s = \int_0^\infty \sigma(E) v(E - U_s(0)) f(v) dv =$

$\langle \sigma v \rangle_s \simeq \langle \sigma v \rangle \exp[-U_s(0)/kT]$ (37)

Plug (34) into (37) to get: $-\frac{U_s(0)}{kT} \simeq \frac{Z_1 Z_2 e^2}{(kT)^{3/2}} (4\pi n_e e^2)^{1/2}$ (38)

$= 0.188 Z_1 Z_2 \zeta \rho^{1/2} T_6^{-3/2}$ where $\zeta^2 = \sum_Z \frac{Z(Z+1)}{A_Z} X_Z$ ($\simeq \frac{X+3}{2}$ if $Z_{tot} \simeq 0$, Pop II)

So, as long as ρ/T^3 is small: $\exp\left(\frac{-U_s(0)}{kT}\right) \simeq 1 + 0.188 Z_1 Z_2 \zeta \rho^{1/2} T_6^{-3/2}$ (39)

This is correction $\lesssim 10\%$ for PP but up to $\sim 50\%$ for CNO ($Z_1 Z_2$ much higher) so must include it.

Homologous Models Including Fusion

Based on: 1) $\epsilon = \epsilon_0 \rho^{a} T^{n}$
2) $\kappa = \kappa_0 \rho T^{-3.5}$
3) $P \propto \rho T$ OR $P \propto T^4$
4) No convection!

Then, if (P, T, ρ, L) are fns(r) for a given M & μ
∃ simple results of scaling to a new M'.

$$\frac{dP}{dr} = \frac{-Gm\rho}{r^2} \qquad \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dT}{dr} = -\frac{3\kappa_0 \rho^2 T^{-3.5}}{16\pi a c r^2 T^3} l \qquad \frac{dl}{dr} = 4\pi r^2 \epsilon_0 \rho^{a+1} T^{n}$$

& $P = \rho \frac{kT}{\mu m_H}$ or $P = \frac{1}{3} a T^4$

Basically the dimensional analysis we performed earlier leads to:

$$\rho_2(x) = \left(\frac{M_2}{M_1}\right)\left(\frac{R_1}{R_2}\right)^3 \rho_1(x) \quad \text{with } x = \frac{r_1}{R_1} = \frac{r_2}{R_2}$$

$$P_2(x) = \left(\frac{M_2}{M_1}\right)^2 \left(\frac{R_1}{R_2}\right)^4 P_1(x)$$

$$T_2(x) = \frac{\mu_2}{\mu_1} \frac{M_2}{M_1} \frac{R_1}{R_2} T_1(x) \quad \text{if gas pressure dominated.}$$

& homologous only if $\frac{\mu_2}{\mu_1} \neq fn(x)$. A full analog requires:

$$L_2(x) = \frac{\epsilon_{02}}{\epsilon_{01}} \left(\frac{\mu_2}{\mu_1}\right)^n \left(\frac{M_2}{M_1}\right)^{n+2} \left(\frac{R_1}{R_2}\right)^{n+3} L_1(x)$$

so $\frac{\epsilon_{02}}{\epsilon_{01}}$ also must be const. in x. Demanding consistency for $\frac{dT}{dr} \Rightarrow \frac{\epsilon_{02}}{\epsilon_{01}} \frac{\kappa_{02}}{\kappa_{01}} \left(\frac{\mu_2}{\mu_1}\right)^{n-7.5} \left(\frac{M_2}{M_1}\right)^{n-3.5} \left(\frac{R_1}{R_2}\right)^{n+2.5} = 1$

If $n=5$ is assumed typical of pp chain (LMS) & $n=18$ for CNO (UMS) & with P_g for LMS & P_{rad} for UMS [i.e., extreme UMS] then:

Lower Main Sequence

$R \sim (K_0 \epsilon_0)^{2/15} (G\mu)^{-1/3} M^{1/5}$

$T \sim (K_0 \epsilon_0)^{-2/15} (G\mu)^{4/3} M^{12/15}$

$L \sim K_0^{-1/6} \epsilon_0^{5/6} (G\mu)^{-4/3} T^{81/12}$

& $L \sim K_0^{-16/15} \epsilon_0^{1/15} (G\mu)^{23/3} M^{27/5}$ or $L \sim M^{5.4}$

Upper Main Sequence

$R \sim (K_0 \epsilon_0)^{3/4} G^{21/164} M^{37/82}$

$T \sim (K_0 \epsilon_0)^{-2/41} G^{5/41} M^{2/41}$

$L \sim K_0^{1/2} \epsilon_0^{3/2} G^{-2} T^{125/4}$ ← steep LHS of H-R diagram!

$L \sim K_0^{-42/41} \epsilon_0^{-1/41} G^{297/164} M^{125/82}$ or $L \sim M^{1.52}$

Real $L \sim M$ is inbetween ⇒ GP more important than RP
but type of reaction also important.

These kinds of considerations yield a strong limit
on G varying w/ time to $\frac{\Delta G}{G} < 10^{-11}\ yr^{-1}$

On LMS dominance of composition through $L \sim \mu^{23/3}$
$T \sim \mu^{4/3} \Rightarrow L \sim T^{23/4}$ near to the LMS obs. $L \sim T^{7}$
∴ H-R diagram is not broadened too much by
composition changes.

For a more general treatment, see
T. R. Carson, Observatory Vol 106, p 71-72 (1986)