Energy Transport

Luminosity of a ☆ is Determined by 2 Independent Conditions:

1) Energy flow rate, ∴ L, det. by ∇T, ∴ energy transport

but

2) L_{static} = rate @ which energy generated by nuclear reactions

Transport
1) Radiative Transfer – what's up now.
2) Convection – if ∇T too big
3) Conduction – important only if degenerate
4) ν losses

Balance: $\epsilon\ (\mathrm{erg\ g^{-1} s^{-1}}) = \epsilon(\rho, T, X)$

If static, then

$$L = \int_V \epsilon \rho \, dV = \int_0^R \epsilon \rho \, 4\pi r^2 dr \qquad (1)$$

or

$$\frac{d\ell(r)}{dr} = \epsilon(r)\,\rho(r)\,4\pi r^2 \quad \text{or} \quad \frac{d\ell}{dm} = \epsilon(m) \qquad (2)$$

If non-static, say expanding, then heat added/g/s = rate of increase of internal energy & rate of expansion work:

$$\frac{dQ}{dt} = \frac{dU}{dt} + P\frac{dV}{dt}$$

If quasi-static, then

$$\frac{dQ}{dt} = T\frac{dS}{dt}$$ & for a shell:

$$\frac{dQ}{dt} = \epsilon(r) - \frac{\ell(r+dr) - \ell(r)}{\rho 4\pi r^2 dr}$$

or

$$\frac{d\ell}{dm} = \epsilon - T\frac{dS}{dt} \qquad (3)$$

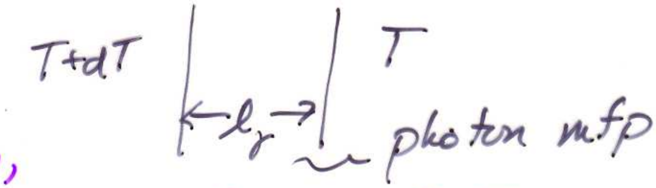
If the gas is monatomic & $P_{rad} \ll P_{gas}$, then

$$S = \frac{N_0 k}{\mu} \ln \frac{T^{3/2}}{\rho} + \text{const} = \frac{3}{2} \frac{N_0 k}{\mu} \ln \frac{P}{\rho^{5/3}} + \text{const}$$

$$\therefore T \frac{dS}{dt} = \frac{3}{2} T \frac{N_0 k}{\mu} \frac{\rho^{5/3}}{P} \frac{d}{dt} \frac{P}{\rho^{5/3}} = \frac{3}{2} \rho^{2/3} \frac{d}{dt} \frac{P}{\rho^{5/3}} \quad \text{or}$$

$$\frac{dl}{dm} = \epsilon - \frac{3}{2} \rho^{2/3} \frac{d}{dt} \frac{P}{\rho^{5/3}}$$

Radiative Transfer



Heat flux, entering−leaving, vacuum is: $F = \sigma (T+dT)^4 - \sigma T^4 \simeq 4 \sigma T^3 dT$

$\therefore$ In a ✶, energy flux/area is: $F = -4 l_\gamma \sigma T^3 \frac{dT}{dx}$ (4)

Recall that we define K as the opacity, or mass absorption coefficient, then the reduction in flux is:

$$\frac{dI}{dx} = -K \rho I$$

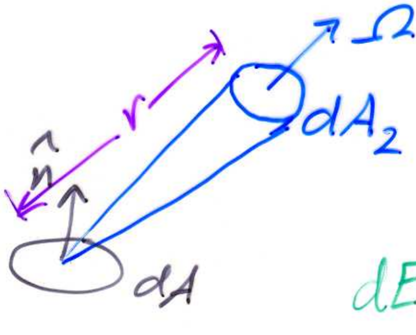
Over short enough distances, $\ni K$ & ρ are constant:

$$I \propto e^{-K\rho x} \Rightarrow l_\gamma \simeq \frac{1}{K\rho}.$$

Since the emission const. for a blackbody is $\sigma = \frac{ca}{4}$ we conclude that (4) $\Rightarrow$

$$F \simeq -\frac{ac}{K\rho} T^3 \frac{dT}{dx} \qquad \text{(off by factor of } \tfrac{4}{3})$$

Radiation Field



$I_\nu(\vec{r}, \Omega, t)$ = monochromatic intensity of field @ $\vec{r}$ @ t

erg cm^{-2} s^{-1} ster^{-1} Hz^{-1}

$dE_\nu = I_\nu(\vec{\Omega}) \, d\Omega \, dt \, d\nu \, \hat{n} \cdot \vec{\Omega} \, dA$ is energy flow from dA to dA_2 in time dt

$$dE_\nu = \frac{I_\nu(\vec{\Omega})}{c} d\Omega \, d\nu \, dV \qquad (5)$$

as $dt = \frac{dl}{c}$, $dV = \hat{n} \cdot \vec{\Omega} \, dA \, dl$

$\therefore$ Energy density (small ΔV out of integral) from integrating (5) over solid angle & volume & divide by volume:

$\Rightarrow u_\nu = \frac{1}{c}\int I_\nu(\Omega)\,d\Omega$ erg cm^{-3} Hz^{-1} is energy density

$p_\nu = \frac{E_\nu}{c}$ (momentum) $\quad \therefore \frac{I_\nu}{c}$ = momentum flux in beam direction

Component $\perp dA = I_\nu \cos\theta/c$ & effective area $= dA\cos\theta$

$\therefore dP_\nu = \frac{I_\nu}{c}\cos^2\theta$ ~ mom. flux/time/area/solid angle/Hz

$$\therefore P_\nu = \int \frac{I_\nu}{c}\cos^2\theta\, d\Omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \mu^2 d\mu \equiv \underbrace{\frac{4\pi}{c} K_\nu}_{2^{nd}\text{ moment}} \qquad (6)$$

P_ν: dyne cm^{-2} Hz^{-1} N.B. $\perp$ to dA if $\mu = \cos\theta$ — anisotropic

Mean intensity: $J_\nu = \frac{1}{4\pi}\int I_\nu(\vec{\Omega})\,d\Omega$ ~ 0^{th} moment of intensity

If isotropic, $J_\nu = I_\nu$

In interiors we often want, $I(\Omega) = \int_0^\infty I_\nu(\Omega)\,d\nu = \int_0^\infty I_\lambda(\Omega)\,d\lambda$

If isotropic, $K_\nu = \frac{1}{3}I_\nu = \frac{c}{4\pi}P_\nu$ $\qquad (7)$

Also " " $u_\nu = \frac{4\pi I_\nu}{c} \Rightarrow \begin{cases} P_\nu = \frac{1}{3}u_\nu \\ P = \frac{1}{3}u \end{cases}$ N.B. no thermal $\rightleftharpoons$ needs to be assumed

$dE \equiv$ energy transfer through $dA = dt\,dA\int I_\nu(\Omega)\,\hat{n}\cdot\vec{\Omega}\,d\Omega\,d\nu$

Radiation flux $\mathcal{F}_\nu$ through $dA = \frac{dE}{dA\,dt\,d\nu}$ or $\mathcal{F}_\nu = \int I_\nu(\Omega)\,\hat{n}\cdot\vec{\Omega}\,d\Omega$

can also define $F_\nu = \frac{\mathcal{F}_\nu}{\pi} = 2\int_{-1}^{1} I_\nu(\mu)\,\mu\,d\mu$ erg cm^{-2} s^{-1} Hz^{-1} ~ 1^{st} moment

Approx. analysis can always be given in terms of I, J, F & K but note that polarization should be included in a more general treatment.

Opacity & Emissivity

Volume opacity or total extinction coef = prob/unit path length that a photon is absorbed or scattered

$k_\nu = k_\nu^a + k_\nu^s$ (cm^{-1}) & $l_\nu = k_\nu^{-1}$ is m.f.p.

Specific opacity: $K_\nu \equiv \frac{k_\nu}{\rho}$ & cross section: $\sigma_\nu = \frac{K_\nu \rho}{n_{abs}}$

Energy lost from a beam in distance dx is:

$$dI_\nu(\Omega) = -I_\nu(\Omega)\, k_\nu\, dx \equiv -I_\nu(\Omega)\, d\tau_\nu$$

$$d\tau_\nu = k_\nu\, dx = \frac{dx}{l_\nu} \text{ is optical depth}$$

$\tau_\nu \gg 1 \Rightarrow$ optically thick $\qquad \tau_\nu \ll 1 \Rightarrow$ optically thin, negligible

Then:
$$I_\nu(x) = I_\nu(0) \exp\left(-\int_0^x k_\nu(x')\,dx'\right) = I_\nu(0)\exp(-\tau_\nu(x)) \qquad (8)$$

w/ $\tau_\nu(x) = \int_0^x k_\nu(x')\,dx'$

But matter can also emit: $dI_\nu(\Omega) = j_\nu(\Omega)\,dx$

w/ $j_\nu(\Omega)$ the spontaneous + stimulated emissivity $\left(\frac{erg\ cm^{-3}\ s^{-1}}{ster^{-1}\ Hz^{-1}}\right)$

$$dI_\nu = \int_{\Omega'} k_\nu^s\, I_\nu(\Omega')\,\frac{d\Omega'}{4\pi}\,dx$$

If isotropic,

$$dI_\nu = k_\nu^s J_\nu\, dx \text{ [for scattering source]}$$

In general,
$$dI_\nu(\Omega) = \int_{\nu'}\int_{\Omega'} k^s(\nu,\Omega,\nu';\Omega')\, I_{\nu'}(\Omega')\,\frac{d\Omega'}{4\pi}\,d\nu'\,dx$$

Eqn of Radiative Transfer: $\frac{dI_\nu}{dx} = \left(\frac{dI_\nu}{dx}\right)_{abs} + \left(\frac{dI_\nu}{dx}\right)_{scatt} + \left(\frac{dI_\nu}{dx}\right)_{em}$

or
$$\frac{dI_\nu(\Omega)}{dx} = -(k_\nu^s + k_\nu^a)\, I_\nu(\Omega) + \int k^s(\nu,\Omega,\nu';\Omega')\, I_{\nu'}(\Omega')\,\frac{d\Omega'}{4\pi}\,d\nu' + j_\nu(\Omega)$$
$$= -k_\nu I_\nu(\Omega) + j_\nu(\Omega) \qquad (9)$$

includes scattering

Convective derivative:

$$\frac{dI_\nu}{dx} = \left(\frac{\partial I_\nu}{\partial t} + v^i I_{\nu,i}\right)\frac{dt}{dx} = \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\beta}\cdot\nabla I_\nu \qquad (10)$$

with $\frac{dx}{dt} = c \Rightarrow \frac{dx^i}{dt} = c\beta^i$

Then, using $d\tau_\nu = k_\nu\,dx$ & dividing by $k_\nu \Rightarrow$ if in <u>steady-state</u>

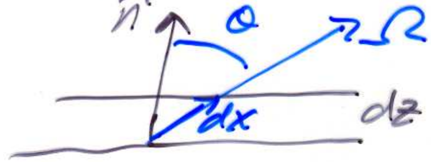
$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{k_\nu} = -I_\nu + S_\nu \qquad (11)$$

source fn.

Multiply by e^{τ_ν} & say $\tau_\nu = 0$ @ beginning; this $\Rightarrow$

$$I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + \int_0^{\tau_\nu} \frac{j_\nu}{k_\nu} \exp(\tau_\nu' - \tau_\nu)\, d\tau_\nu'$$

For a const. radiation field: $\frac{dI_\nu}{dt} = 0$ & if axially symmetric then (11) becomes



$$\cos\theta\, \frac{dI_\nu}{dz} = -k_\nu I_\nu + j_\nu$$

or
$$-\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu \qquad (12)$$

Integrate over $d\Omega = \sin\theta\, d\theta\, d\varphi$ & assume axisymmetry

$$-\frac{d}{d\tau_\nu}\int_{-1}^{1} \mu I_\nu\, d\mu = \int_{-1}^{1} I_\nu\, d\mu - \int_{-1}^{1} S_\nu\, d\mu \qquad \text{or}$$

$$-\frac{1}{2}\frac{dF_\nu}{d\tau_\nu} = 2J_\nu - \int_{-1}^{1} S_\nu\, d\mu$$

If source fn. has no preferred direction, then S_ν indep. of μ

$$\Rightarrow\ -\frac{1}{4}\frac{dF_\nu}{d\tau_\nu} = J_\nu - S_\nu \qquad (13)$$

used in stellar atms. Take ~~2nd~~ 1st moment of (12) & integrate

$$-\frac{d}{d\tau_\nu}\int_{-1}^{+1} \mu^2 I_\nu\, d\mu = \int_{-1}^{1} \mu I_\nu\, d\mu - \int_{-1}^{1} \mu S_\nu\, d\mu \qquad \text{or}$$

$$-2\frac{dK_\nu}{d\tau_\nu} = \frac{F_\nu}{2} - \int_{-1}^{+1} S_\nu \mu\, d\mu$$

If <u>isotropic</u> again, then: $-\frac{dK_\nu}{d\tau_\nu} = \frac{1}{4}F_\nu$

or
$$\pi F_\nu = -4\pi \frac{dK_\nu}{d\tau_\nu} = -c\,\frac{dP_\nu}{d\tau_\nu} \qquad (14)$$

Flux $\propto -\nabla P_{rad}$

Integrate this over freq. for total energy flux, needed for stellar interiors

$$\int_0^\infty \pi F_\nu\, d\nu = -\int_0^\infty \frac{c}{k_\nu}\frac{dP_\nu}{dx}\, d\nu$$

If we define the mean (over ν) opacity as:

$$\bar{k} = \frac{\int_0^\infty \frac{dP_\nu}{dx} d\nu}{\int_0^\infty \frac{1}{k_\nu} \frac{dP_\nu}{dx} d\nu}, \quad \text{then ; as } P_{rad} = \int_0^\infty P_\nu \, d\nu$$

$$\pi F = -\frac{c}{\bar{k}} \frac{dP_{rad}}{dx} \qquad (15)$$

But note that if radiation is very intense, then it does work on the gas & additional terms are needed.

In the stellar interior, the radiation field is closely matched to black body : the Planck fn $\Rightarrow$ LTE here

$$I_\nu \equiv B_\nu(T) = \frac{2h\nu^3}{c^2} \left[\exp(h\nu/kT) - 1 \right]^{-1}$$

It is easy to show $\quad J_\nu = B_\nu(T) \qquad K_\nu = \frac{1}{3} B_\nu(T)$

$$u_\nu = \frac{4\pi}{c} B_\nu(T) \qquad F_\nu = 0 \qquad P_\nu = \frac{4\pi}{3c} B_\nu(T)$$

& $B(T) = \int_0^\infty B_\nu(T) d\nu = \dfrac{2k_B^4 \pi^4}{15 h^3 c^2} T^4 = \dfrac{\sigma}{\pi} T^4 = \dfrac{uc}{4\pi}$

$\therefore$ Inside a ☆ $\quad P_\nu = \dfrac{4\pi B_\nu(T)}{3c}$

Insert this into (15) and integrate over frequency

$$\pi F = -\frac{4\pi}{3} \int_0^\infty \frac{1}{k_\nu} \frac{dB_\nu}{dT} \frac{dT}{dx} d\nu$$

Define the Rosseland mean opacity :

$$\frac{1}{\bar{k}} \equiv \frac{\int_0^\infty \frac{1}{k_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} \qquad (16)$$

Since $\dfrac{dB_\nu}{dT} = \dfrac{4\sigma T^3}{\pi}$ & since $\dfrac{dB_\nu}{dT}$ peaks $\sim 4kT$ heat mainly carried by above average energy photons

$$\pi F = -\frac{4ac}{3\bar{k}} T^3 \frac{dT}{dx} \qquad (17)$$

N.B. Flux driven by $\nabla T \Leftrightarrow \mathcal{O}(10^{-11})$ deviation from LTE

Since $\ell(r) \equiv 4\pi r^2 \pi F$ we get:

$$\ell(r) = -\frac{16\pi ac}{3\bar{k}} r^2 T^3 \frac{dT}{dr} \qquad (18)$$

is the relation between ℓ & $\frac{dT}{dr}$ in a stellar interior.

Under these circumstances we can also write:

$$aT^4 = u \quad \therefore \quad 4aT^3 \frac{dT}{dr} = \frac{du}{dr} \quad \text{\& from (17)}$$

$$\mathcal{F} = -\frac{c}{3\bar{\kappa}}\frac{du}{dr} = -\underbrace{D}_{\text{diffusion coef.}}\frac{du}{dr}$$

Radiative Acceleration: $\dfrac{\bar{\kappa}\ell}{4\pi r^2 c} = -\dfrac{d}{dr}\dfrac{1}{3}aT^4 = -\dfrac{dP_{rad}}{dr}$ (19)

∴ If radiation accel is a_r & $f_r = -\dfrac{dP_{rad}}{dr}$, then

$a_r = \dfrac{\bar{\kappa}\,\ell(r)}{4\pi r^2 c\rho} = \dfrac{\bar{K}\,\ell(r)}{4\pi r^2 c}$ Demanding $a_r \le a_{grav} = \dfrac{Gm(r)}{r^2} \Rightarrow$

$$\ell(r) = \frac{4\pi c G M_\odot}{\bar{K}}\left(\frac{m(r)}{M_\odot}\right) \Rightarrow L_{Edd} = \frac{5.0\times10^{37}}{\bar{K}}\left(\frac{M}{M_\odot}\right)\ \text{erg s}^{-1} \quad (20)$$

If electron scattering dominates opacity: $\bar{K} \simeq 0.2(1+X)$

or $L_{Edd} \simeq 1.4\times10^{38}\left(\dfrac{M}{M_\odot}\right)$ erg s^{-1} ⇒ A Limit on Most Massive ☆s

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Strömgren's Theorem Rewrite (19) as:

$$\frac{dP_{rad}}{dr} = \frac{-K\rho}{4\pi c r^2}\ell(r) \quad (21)$$

If the ☆ is static: $\dfrac{d\ell(r)}{dr} = 4\pi r^2\rho\epsilon$ & $\dfrac{dP}{dr} = \dfrac{-Gm(r)}{r^2}\rho$ —

divide into (21) to get: $\dfrac{dP_r}{dP} = \dfrac{K\ell(r)}{4\pi c G M(r)}$

Let $\eta \equiv \dfrac{\bar{\epsilon}_r}{\bar{\epsilon}} = \dfrac{\ell(r)/m(r)}{L/M}$ the ave. rate of en. gen. interior to r / ave. rate in entire ☆

$$\therefore \frac{dP_r}{dP} = \frac{L}{4\pi c G M}K\eta \quad \text{or} \quad P_r(r) = \frac{L}{4\pi c G M}\int_0^{P(r)} K(r)\eta(r)\,dP = \frac{L}{4\pi c G M}P(r)\,\overline{K\eta} \quad (22)$$

pressure ave. bet r & R

The ratio of the rad. pressure to total pressure @ a pt. w/in a star in radiative $\rightleftarrows$ $\propto \overline{K\eta}$ for points exterior to r, w/ ave. taken w.r.t. dP where P is the total pressure

Applications 1) @ center ⇒ $L = \dfrac{4\pi c G M(1-\beta_c)}{\overline{K\eta}}$ ~ entire ☆ ; $\beta_c = \dfrac{P_{gas}}{P_{tot}}$

2) $K\eta = $ const ⇒ $\overline{K\eta(r)} = $ const ⇒ $\beta(r) = $ const

∴ Radiative ☆ w/ fixed β ⇒ polytrope of n=3 a standard model.

In real ☆s K rises from center to surface while η falls a lot — Compensation ⇒ n=3 is decent model. Note you can prove n=3 polytrope, that $T_c \propto M^{2/3}\bar{\rho}^{1/3}$ but $\bar{\rho} \propto M/R^3$ ∴ $T_c \propto M/R$

If $\dfrac{dT}{dR} \propto \dfrac{T_c}{R}$ then if $\bar{K} \propto M^0$ (indep), $L \propto R^2 \dfrac{(M/R)^3}{M/R^3}\dfrac{M/R}{R} \propto M^3$ — close to MS relation!