Sources of Opacity: Need to compute the Rosseland mean opacity $\kappa = \kappa(\rho, T, X)$ – a key constitutive relation

In stellar interiors ∃ just four dominant processes:

1) Bound-bound — true absorption of photons: – photo excitation [Inverse – photoemission]
2) Bound-free — photoionization – true-abs [Inv.: radiative-recombination]
3) Free-free abs. — abs. of γ by a continuum e^- as it passes an ion & gains energy – true-abs [Inv.: bremsstrahlung]
4) Scattering from free-electrons – Compton (or Thomson) scattering – Not true absorption, as scattered energy ≃ incident en. in ★s – But lengthens time γ's in ★ so ⇒ other true-abs can act more [Inv: Inverse Compton]

In outer layers & photosphere its cool enough so that:

5) Atomic lines (b-b) & even
6) Molecular lines can contribute

Correct treatment involves time-dependent perturbation theory. We will not go through the computations here – good place to begin is § 3-3 of Clayton.

Photons are degraded as they diffuse from a stellar core towards the surface, and since $l_\gamma \ll R$ ∃ many collisions to produce the tremendous drop, from MeV to eV in core ⇒ photospheric photons.

Process i ⇒ $\sigma_i(\nu)$ — the cross-section for absorption or scattering:

$$\kappa_i(\nu) = \frac{n_i \sigma_i(\nu)}{\rho} \quad \text{cm}^2\,\text{g}^{-1}$$

$$\therefore \kappa_{tot}(\nu) = \sum_i \kappa_i(\nu)$$

$$\frac{1}{\kappa} = \langle \kappa_{tot}(\nu)^{-1} \rangle = \left\langle \left[\sum_i \kappa_i(\nu) \right]^{-1} \right\rangle \simeq \frac{1}{\kappa_j} \qquad (23)$$

where the j^{th} process dominates

Bound-bound – brief summary

A_{nm} – Einstein A coef.: Prob/time of spontaneous transition from n down to m

$u_\nu B_{mn} = B_{mn} \underbrace{\frac{8\pi}{c^3} \frac{h\nu^3}{e^{h\nu/kT}-1}}_{\text{if LTE}}$ – prob. that atom will absorb appropriate photon from continuum w/ $\nu = (E_n - E_m)/h$

Detailed balancing demands that:

$$N_n A_{nm} + \underbrace{N_n u_\nu B_{nm}}_{\text{induced em.}} = N_m u_\nu B_{mn} \quad (24)$$

~ rate of excitation = rate of deexcitation

Solve for $u_\nu = \dfrac{A_{nm}/B_{nm}}{\frac{N_m}{N_n}\frac{B_{mn}}{B_{nm}} - 1}$ and compare w/ LTE rad. energy-density formula $\Rightarrow$

$$g_m B_{mn} = g_n B_{nm}, \quad A_{nm} = \frac{8\pi h\nu^3}{c^3} B_{nm} \quad (25)$$

& $\dfrac{N_m}{N_n} = \dfrac{g_m}{g_n} e^{h\nu/kT}$ with the g's the statistical weights.

Natural line broadening can be given a semi-classical treatment and: $I(\nu) = I_0 \dfrac{\gamma}{4\pi(\nu-\nu_0)^2 + (\gamma/2)^2}$

w/ $\dfrac{\gamma}{2\pi}$ the FWHM of the line. $A \approx \gamma$

In particular $\exists$ oscillator strengths, the matrix elements.

$$f_{nm} = \frac{2}{\pi m_e \omega_{nm}} \left| \langle n | \exp\left(i \frac{\omega_{nm}}{c} \hat{n}\cdot\vec{r}\right) \vec{p}\cdot\vec{e} \, | m \rangle \right|^2$$

($\hat{n}\cdot\vec{r}$: momentum; $\vec{p}\cdot\vec{e}$: polarization direction)

and $A_{nm} = 3 \dfrac{g_m}{g_n} f_{nm} \gamma = \dfrac{g_m}{g_n} \dfrac{8\pi^2 e^2 \nu^2}{m_e c^2} f_{nm}$ (26)

For Hydrogen, Kramer's formula $\Rightarrow$

$$f_{nm} = \frac{2^6}{3\sqrt{3}\,\pi} \frac{1}{g_m} \frac{1}{\left(\frac{1}{m^2} - \frac{1}{n^2}\right)^3} \frac{g_I}{n^3 m^3}$$

g_I ~ Gaunt factor for b-b transition

To compute the actual cross-section, note that we define

$\sigma_\nu \Rightarrow P$ (absorption/unit path length) $= N\sigma_\nu$ ← # density

Energy taken from beam $= I_\nu N\sigma_\nu c\,dt$

transitions: $I_\nu N\sigma_\nu \frac{dt}{h\nu}$ & # trans./vol/time/freq $= I_\nu \sigma_\nu N/h\nu$

Integrate over $d\Omega$ & assume in LTE $\Rightarrow I_\nu = \frac{cu_\nu}{4\pi}$

$\Rightarrow$ # abs $= cu_\nu N\sigma_\nu/h\nu$

& trans. rate $= cN\int_0^\infty \frac{u_\nu}{h\nu}\sigma_\nu\,d\nu = u_{\nu_0} B_{nm} N$

(trans/vol/time) Since $\sigma_\nu \simeq 0$ except very close to $\nu_0 = \frac{E_n - E_m}{h}$ we pull $\frac{u_\nu}{h\nu}$ out of the $\int \Rightarrow$

$$u_{\nu_0} B_{nm} N \simeq \frac{cu_{\nu_0} N}{h\nu_0}\int_0^\infty \sigma_\nu\,d\nu$$

Use (26) & (25) to $\Rightarrow$ $\int_0^\infty \sigma_\nu\,d\nu = \frac{\pi e^2}{m_e c} f_{nm}$

$= 2.65\times10^{-2} f_{nm}$ cm² Hz

Using $\sigma_\nu = \frac{\pi e^2}{m_e c} f_{nm}\phi_\nu$ ~ $\int \phi_\nu d\nu = 1$ — the line broadening fn.

$\therefore \sigma_\nu = \sigma_0 \frac{\gamma}{4\pi^2(\nu-\nu_0)^2 + (\gamma/2)^2}$ and finally,

$$\sigma_\nu = \frac{\pi e^2}{m_e c} f_{nm} \frac{\gamma}{4\pi^2(\nu-\nu_0)^2 + (\gamma/2)^2} \qquad (27)$$

In general, γ dominates $\gamma_{natural} \propto f_{nm}$

$\therefore$ @ the line center, $\nu = \nu_0$: $\sigma_\nu = \frac{4\pi e^2}{m_e c}\frac{f_{nm}}{\gamma}$

and for a natural line broadening: $\gamma \simeq A \simeq 3 f_{nm}\gamma_{ce} \Rightarrow f_{nm}/\gamma \simeq \frac{1}{3\gamma_{ce}}$

or $\sigma_\nu \simeq \frac{4\pi e^2}{m_e c} 3 \frac{3 m_e c^3}{8\pi^2 e^2}\frac{1}{\nu_0^2} = \frac{9}{2\pi}\frac{c^2}{\nu_0^2} = \frac{9}{2\pi}\lambda_0^2$

We ought to correct the above for stimulated emission — crudely this is negative absorption:

Net # of abs $= N_m u_\nu B_{mn} - N_n u_\nu B_{nm} = N_m u_\nu B_{nm}\left(1 - \frac{N_n B_{nm}}{N_m B_{mn}}\right)$

$\therefore \sigma_\nu = (1 - e^{-h\nu/kT})\frac{\pi e^2}{m_e c} f_{nm}\phi_\nu$ $= N_m u_\nu B_{nm}(1 - e^{-h\nu/kT})$

& $k_{lines}(\nu) = n\sigma_\nu$ ← # density of absorbing atoms

Electron Scattering

A classical treatment for low energy scattering is good enough since $\lambda_{\star} \gg \lambda_c = \frac{\hbar}{m_e c} = 2.4\times10^{-10}$ cm

An EM wave of strength $\mathcal{E} \Rightarrow a = \frac{e\mathcal{E}}{m_e}$

Energy that can be absorbed, *in toto* is:

$$\frac{dE}{dt} = \frac{2}{3}\frac{e^2}{c^3}a^2 \qquad (28)$$

$$\therefore \left\langle \frac{dE}{dt} \right\rangle = \frac{2}{3}\frac{e^2}{c^3}\frac{e^2}{m_e^2}\langle \mathcal{E}^2 \rangle$$

But the Poynting flux (or incoming energy flux) is: $\frac{c}{4\pi}\langle \mathcal{E}^2 \rangle$

& as: $\sigma = \frac{\left\langle \frac{dE}{dt} \right\rangle_{scatt}}{\frac{dE}{dt}\Big|_{inc}} \Rightarrow \sigma_T = \frac{\frac{2}{3}\frac{e^2}{c^3}\frac{e^2}{m_e^2}\langle \mathcal{E}^2 \rangle}{\frac{c}{4\pi}\langle \mathcal{E}^2 \rangle}$

or

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65\times10^{-25}\ \text{cm}^2 \qquad (29)$$

$\therefore \kappa_{es} = \frac{\sigma_T n_e}{\rho} = 0.2(1+X)$ Dominates κ for $\star$s w/ $M \gtrsim 3M_\odot$

This is nicest, since it's isotropic & *not* fn(ν)!

Rayleigh Scattering: A low energy photon scattered off a bound atomic electron [*not* regular bound-bound]

Classical treatment also OK: if $\hbar\omega / E < \Delta E$ transition is absorbed & e then oscillates about normal energy level:

$$a = \frac{e\mathcal{E}}{m} - \omega_0^2 r$$

Similar arguments lead to: $\sigma = \sigma_T \frac{1}{[1-(\omega_0/\omega)^2]^2}$,

or for $\hbar\omega \ll \hbar\omega_0$, the restoring force of the harmonic oscillator.

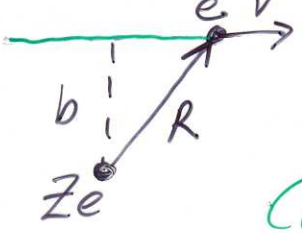
$$\sigma_{Rayl} \simeq \sigma_T \left(\frac{\omega}{\omega_0}\right)^4 = \sigma_T \left(\frac{\lambda_0}{\lambda}\right)^4$$

This relies on $kT \ll \Delta E \lesssim 5\times10^3$ K, typically.

$\therefore$ Rayleigh scattering is important in stellar & planetary atmospheres, but NOT in stellar interiors.

"Why is the sky blue, Mommy?" "Rayleigh scattering, my dear."

Free-free – Bremsstrahlung – Dominates in Low Mass Stellar Interiors



An approximate result is obtained for an e moving in the field of an ion which absorbs or emits a photon

Conservation of energy demands

$$\frac{1}{2} m_e v_f^2 + h\nu = \frac{1}{2} m_e v_i^2$$

We need the acceleration to plug into (28) — make the impulse approximation that $\Delta t \simeq \frac{2b}{v_i}$ and $a \simeq \frac{Ze^2}{m_e b^2}$ so

$$E_{abs/ion/e} = \int_{-\infty}^{\infty} \frac{dE}{dt} dt = \frac{2}{3}\frac{e^2}{c^3} \int_{-\infty}^{\infty} \frac{Z^2 e^4}{m_e^2 b^2} dt \simeq \frac{2}{3}\frac{Z^2 e^6}{m_e^2 c^3}\frac{1}{b v_i} \quad (30)$$

We need the frequency dependence. ∴ Take Fourier trans:

$$\frac{dE(\nu)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dE(t)}{dt} e^{-2\pi i \nu t} dt$$

Clearly only if $2\pi\nu t \approx 1$ while $t \simeq \frac{b}{v_i}$ will there be significant contributions, as otherwise the integral tends to cancel $\Rightarrow 2\pi\nu \simeq \frac{v_i}{b}$

∴ $dW_\nu = 2\pi b\, db\, E_{abs}$ — The energy abs/e/ion interval $d\nu$

Use (30) $$= \frac{4\pi Z^2 e^6}{3 m_e^2 c^3 v_i}\frac{2\pi d\nu}{v_i} = \frac{8\pi^2 Z^2 e^6}{3 m_e^2 c^3 v_i^2} d\nu \quad (30a)$$

But the total energy absorbed requires: n_i, flux of e's of velocity $(v_i) = n_e v_i f(v_i) dv_i$ & dw_ν, or

$$\left.\frac{dW_\nu}{dV}\right|_{em} = n_i n_e f(v_i) \frac{8\pi^2 Z^2 e^6}{3 m_e^2 c^3 v_i} dv_i d\nu \quad (31)$$

For absorption onto e's of v_f going to v_i ∃ the cross-section σ_ν: abs/ion/e of vel. v_f from a radiation field of energy density u_ν.

If in LTE: $u_\nu = \frac{4\pi}{c} B_\nu(T)$, so the net energy abs. is:

$$\left.\frac{dW_\nu}{dV}\right|_{abs} = n_i n_e f(v_f) dv_f \sigma_\nu \cdot \underbrace{\left(1 - e^{-h\nu/kT}\right)}_{\text{stimulated emission}} \cdot c u_\nu d\nu \quad (32)$$

But again, if in thermal $\rightleftharpoons$ detailed balance demands: (31)=(32)

or: $n_i n_e f(v_i) \frac{8\pi^2 Z^2 e^6}{3 m_e^2 c^3 v_i} dv_i d\nu = n_i n_e f(v_f) dv_f \sigma_\nu c \frac{8\pi h\nu^3}{c^3} (e^{h\nu/kT} - 1)^{-1} (1 - e^{-h\nu/kT}) d\nu$

or $\frac{\pi Z^2 e^6}{3 m_e^2 v_i} \cdot f(v_i) dv_i = \frac{c h \nu^3}{e^{h\nu/kT}} \sigma_\nu f(v_f) dv_f$

Maxwellian: $f(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(\frac{-mv^2}{2kT}\right) 4\pi v^2 dv$

But $\therefore \frac{f(v_f)}{f(v_i)} = \exp\left[\frac{-m_e}{2kT}(v_f^2 - v_i^2)\right] \frac{v_f^2}{v_i^2}$ or

$$\sigma_\nu = \frac{\pi}{3} \frac{Z^2 e^6}{h c m_e^2 \nu^3 v_i} \frac{v_i^2}{v_f^2} \cdot \exp\left[\frac{-1}{kT}\left(\underbrace{\frac{m}{2} v_f^2 + h\nu - \frac{m}{2} v_i^2}_{0}\right)\right]$$

So, crudely, $\sigma_\nu = \frac{4\pi}{3\sqrt{3}} \frac{Z^2 e^6}{h c m_e v \nu^3}$ (33)

when the difference between v_i & v_f is properly accounted for

The total bremsstrahlung opacity is:

$$K_{ff}(\nu) = \int_0^\infty n_i n_e f(v) \sigma_\nu dv \, g_{ff}(\nu, v)$$

Gaunt factor to account for quantum vs classical calc

$$K_{ff}(\nu) = \frac{4}{3}\left(\frac{2\pi}{3 m_e kT}\right)^{1/2} \frac{Z^2 e^6}{h c m_e \nu^3} n_i n_e \overline{g_{ff}}(\nu) \quad (34)$$

Thermal averaged Gaunt factor.

Take the leading dependence of $K_{ff}(\nu) \propto \nu^{-3} T^{-1/2}$ & substitute into the definition of the Rosseland mean opacity to get:

$$\overline{K_{ff}} = K_0 \rho T^{-3.5} \quad \text{with} \quad (35)$$

$$K_0 = 3.7 \times 10^{22} (1+X)(1-Z) \overline{g_{ff}}$$

Kramers' opacity.

Bound-free

Excitation from bound to ionized state — most important in atmospheres, but a real contributor in interiors too!

For a hydrogenic atom, i.e. one electron in a shell:

$$E_n = -\frac{m_e Z^2 e^4}{2\hbar^2 n^2} \equiv -\frac{I_H Z^2}{n^2} \quad (36)$$

(I_H: H ionization potential)

$\therefore \frac{1}{2} m_e v^2 - E_n = h\nu$ is energy conservation (37)

This is similar to bremsstrahlung, except going to a discrete state. A semi-classical treatment notes that most of the e's energy (in capture) is radiated before capture, so $\therefore$ en. loss/continuum $\approx$ (30a)

Then cross-section for emission of photons into the interval $d\nu$ is

$$\boxed{d\sigma_\nu = \frac{dw_\nu}{h\nu}} \quad (38a)$$

To allow for discreteness, note that to capture to state E_n, we define

$$\boxed{d\sigma_\nu \equiv \sigma_{cn}\, dn} \quad (38b)$$

So $h\nu \frac{d\sigma_\nu}{d\nu} = \frac{dw_\nu}{d\nu} = h\nu\, \sigma_{cn} \frac{dn}{d\nu}$

$$\therefore \sigma_{cn} = \frac{1}{h\nu} \frac{dw_\nu}{d\nu} \left(\frac{dn}{d\nu}\right)^{-1}$$

(37) $\Rightarrow$ $-E_n = \frac{m_e Z^2 e^4}{2\hbar^2 n^2} = h\nu - \frac{1}{2} m_e v^2$ so

$$\left(\frac{dn}{d\nu}\right)^{-1} = \frac{m_e Z^2 e^4}{\hbar^2 h} n^{-3}$$

$$\therefore \sigma_{cn} = \frac{4\pi^2 m_e Z^2 e^4}{h^4 \nu n^3} \cdot \underbrace{\frac{8\pi^2 Z^2 e^6}{3 m_e^2 c^3 v_i^2}}_{(30a)} = \frac{32}{3}\pi \frac{4 Z^4 e^{10}}{m_e c^3 h^4 v^2 \nu n^3} \quad (39$$

Note, this cross-section is for capture into state n but photoionization is related by detailed balance

If the actual photoionization cross-section is $\sigma_{\nu n}$, then # of photons of $h\nu$ abs/s w/ em. of e's of energy $\frac{1}{2}m_e v^2 - E_n$ from the n^{th} state is:

$$\frac{cu_\nu}{h\nu} d\nu \, \sigma_{\nu n} N_n \left(1 - e^{-h\nu/kT}\right) \quad (40)$$

(N_n: # of atoms in state n; $1 - e^{-h\nu/kT}$: spontaneous emission)

Captured e's, w/ velocity v, into state n/sec is:

$$N_i \, \sigma_{cn} \, n_e \, v \, f(v) \, dv \quad (41)$$

∴ Can equate these to obtain:

$$\sigma_{\nu n} = \frac{h\nu N_i n_e v f(v) dv}{c u_\nu N_n \left(1 - e^{-h\nu/kT}\right) d\nu \, \sigma_{cn}} \quad (42)$$

But Boltzman statistics demand (for H)

$$\frac{N_n}{N_i} = \frac{g_n}{g_1} \exp\left[-I_H Z^2 \left(1 - \frac{1}{n^2}\right)/kT\right] \quad (43a)$$

and Saha eqn says:

$$\frac{N_i N_e}{N} = 2 \frac{g_i}{g} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-I_H Z^2/kT} \quad (43b)$$

(N: total # of atoms)

Since $\frac{N}{N_i} = \frac{g}{g_1}$ w/ $g = \sum_n g_n \exp\left[-I_H Z^2 \left(1 - \frac{1}{n^2}\right)/kT\right]$, the partition function, we get

$$\sigma_{\nu n} = \frac{g_i}{g_n} \left(\frac{m_e v c}{h\nu}\right)^2 \sigma_{cn} \quad (44)$$

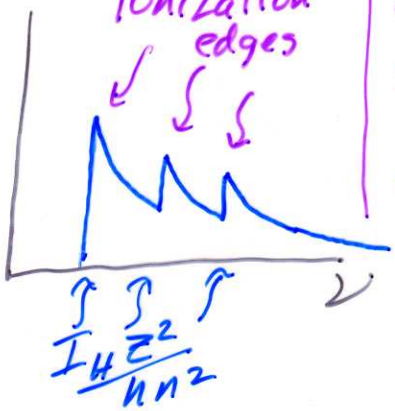
Note that $g_1 = 2$ & $g_n = 2n^2$ for hydrogen-like atoms. But these expressions have assumed hydrogenic atoms, so $Z-1$ e's don't participate.

⇒ $g_i = 1$, & w/ Gaunt factor to take into account Q-M effects:

$$\sigma_{\nu n} = \frac{64\pi^4}{3\sqrt{3}} \frac{m_e Z^4 e^{10}}{h^6 c \nu^3 n^5} g_{bf}(\nu, n) \qquad (45)$$

$g_{bf} \approx 1$

Note that $\sigma_{\nu n} = 0$ if $h\nu < I_H Z^2/n^2$, or



for atomic states w/ $n > \left(\frac{I_H Z^2}{h\nu}\right)^{1/2} \equiv n_*$

$\Rightarrow$ This fixes the absorption edge

Furthermore, since

$\sigma_{\nu n} \sim (h\nu)^{-3} n^{-5}$ & $h\nu \approx \frac{1}{n^2}$ we get

$\sigma_{\nu n}/\text{peak} \sim n$

For bound-free to be important, many atoms must have $n > n_*$

$$\therefore \kappa_{bf}(\nu) = \sum_{n > n_*} \frac{N_n}{\rho} \sigma_{\nu n} \qquad (46)$$

$\Rightarrow$ Mostly ionized if so many excited, so $N \simeq N_i$

Use (43b) to conclude $N_n \propto n^2$ if $n \gg 1$.

But then, since $\sigma_{\nu n} \sim n$ @ the absorption edge, each term in (46) is like $n^3 \Rightarrow$ most of κ_{bf} is due to highly excited states. Once $h\nu > h\nu_{\text{lowest}}$ ∄ more edges & $\kappa \sim \nu^{-3}$. Now plug this into the Rosseland mean opacity to obtain:

$$\kappa_{bf} = \kappa_0 \rho T^{-7/2} \qquad (47)$$

& $\kappa_0 = 4.3 \times 10^{25} Z(1+X) \frac{\overline{g_{bf}}}{t_{bf}}$ ~frequency averaged Gaunt factor ~1

& t_{bf} is the guillotine factor, which corrects for overestimate in # of bound e's/atom due to approximations needed to get (47). $t_{bf} \approx 10$ is reasonable