Rotating Stars

The key parameter describes the ratio of centrifugal to gravitational force: balance is $\frac{GMm}{r^2} = m\Omega^2 r \Rightarrow \chi = \frac{\Omega^2}{GM/r^3}$

For a <u>uniform sphere</u> a ratio close to this is standard

$$\chi \equiv \frac{\Omega^2}{2\pi G\rho} \qquad (1)$$

Simple models from the 19th century are the <u>McLaurin spheroids</u> (see Chandrasekhar, Ellipsoidal Figures of Equilibrium)

these have $a = b > c$ so $e = \sqrt{1-\left(\frac{c}{a}\right)^2}$ (2)

$e = 0 \Rightarrow$ sphere, stable ($\chi = 0$ for both sphere & infinite disk)

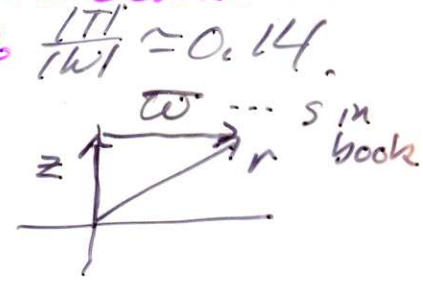
Increase J, the ang. mom., $\Omega\uparrow$, $\chi\uparrow$ & $e\uparrow$ UNTIL $e = 0.9299$.

Then as J continues to rise, $\Omega\downarrow$ as $I\uparrow$ faster than $J\uparrow$

But @ $e = 0.8127$ or $\chi = 0.1868$, McLaurin spheroids bifurcate into Jacobi ellipsoids ($a > b > c$) as the later have lower total (grav + kinetic) energy. $\therefore$ Friction, viscosity or gravitational radiation drive spherords to ellipsoids. This is a secular instability, rather slow, for $\frac{T}{|W|} = 0.1376$. Further, for $\frac{T}{|W|} = 0.1604$, a dynamical instability sets in $\Rightarrow$ pear shapes.

<u>N.B.</u> Realistic rotating models w/ ρ greater @ center still $\Rightarrow$ slow instab to non-axisymmetry @ $\frac{|T|}{|W|} \simeq 0.14$.

<u>Roche model</u> – The opposite approx. often used for binary stars.



All mass @ center, so total potential is:

$$\Psi = \Phi + V = -\frac{GM}{(\varpi^2+z^2)^{1/2}} - \frac{1}{2}\varpi^2\Omega^2 \qquad (3)$$

Equipotential surfaces are given by:

$$z^2 = \frac{1}{(a-b\varpi^2)^2} - \varpi^2 \qquad (4)$$

where $a = \frac{1}{r_p}$ @ $\varpi = 0$ & $b = \frac{\Omega^2}{2GM}$

$r_p \approx$ polar radius

$z=0$ in eq. plane $\Rightarrow \varpi_{cr} = \left(\frac{GM}{\Omega^2}\right)^{1/3}$ where $\nabla\Psi = 0$ (5)

For $\varpi < \varpi_{cr}$ gravity dominates, $\varpi > \varpi_{cr}$ rotation dominates.

$V_{cr} = 0.5412 \frac{4\pi}{3}\varpi_{cr}^3$ from numerical integration.

If a ☆ fills the critical (Roche) volume, it has $\bar{\rho} = \frac{M}{V_{cr}}$

& redefine : $\chi' \equiv \frac{\Omega^2}{2\pi G\bar{\rho}}$ (1')

for $\frac{\text{cent}}{\text{grav}}$ @ equator. Filling vol. $\rightleftarrows \chi'_{cr} = 0.361$

As one spins up a ☆ w/ M starting below χ'_{cr} it expands, & from (5) $\varpi_{cr} \searrow$ ∴ more oblate until critical surface reached. Then mass is shed — leaves a ring near ϖ_{cr} — if forced to co-rotate it would fly to ∞, but if j is conserved then it hangs around:

$\Omega = \text{const}$: centrifugal $\propto \varpi$ but grav $\propto \frac{1}{\varpi^2}$

$j = \text{const}$: $\Omega \propto \varpi^{-2}$, cent $\propto \Omega^2\varpi \propto \frac{1}{\varpi^3}$ — faster than grav. fall off.

If ∃ a neighboring ☆ potential is modified & the lost mass is partially accreted onto the companion.

Physically more likely is <u>contraction</u> for a rotating ☆: $\Omega \propto R^{-2}$ but $\varpi_{cr} \propto \Omega^{-2/3} \propto R^{4/3}$ so critical surface drops faster than mass losing stellar radius $\Rightarrow$ disk left behind to become proto-planetary nebula, some of which (re)accretes, some condenses, some blown away.

More realistic models can be constructed for slowly rotating polytropes, first by Chandra (K&W 41.3)

The equipotential surfaces can be expanded as:

$$w \equiv \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{1/n} : w = w_0(\xi) + \chi w_1(\xi) + \chi w_2(\xi) P_2(\theta) + \ldots \quad (6)$$

where $w_0(\xi)$ is sol'n of Lane-Emden eqn : $\xi = \left(\frac{4\pi G}{(n+1)K}\right)^{1/2} \rho_c^{\frac{n-1}{2n}} r$

Let $\chi'' \equiv \frac{\Omega^2}{2\pi G\rho_c}$ (1'').

w_1 is the spherically sym. distortion & w_2 is leading non-spherical distortion, w/ P_2 the Legendre polynomial order 2

$$\epsilon \equiv \frac{r_{eq} - r_p}{r_{eq}} = 3.75\,\chi'',\ 9.82\,\chi'',\ 41.8\,\chi'',\ 468\,\chi''$$

$n=1$ $\quad n=2$ $\quad n=3$ $\quad n=4$

{oblateness or ellipticity

<u>Conservative rotation</u> $\exists\, V \ni \Omega^2 \varpi \hat{e}_\varpi = -\nabla V$ (7)

& ang. vel. or ang. mom. depends only on cylindrical radial coordinate: $\frac{\partial \Omega}{\partial z} = \frac{\partial \Omega}{\partial \varphi} = 0$ i.e. constant on cylinders or pseudo-barytropic model. Then:

$$V = -\int_0^{\varpi} \Omega^2 \varpi\, d\varpi \quad ; \quad \Psi = \Phi + V \qquad (8)$$

Then the eqn. of hydrostatic $\rightleftarrows$ becomes:

$$\nabla P = -\rho \nabla \Psi \qquad (9)$$

Already noted $\Rightarrow \nabla P$ & $-\nabla\Psi$ $\parallel$ or $\Psi = \text{const} \Leftrightarrow$ isobaric equipot. surfaces or $P = P(\Psi)$ $\therefore \rho = \frac{dP}{d\Psi} = \rho(\Psi)$. <u>If</u> ideal gas: $\frac{T}{\mu} = \frac{P}{\rho \mathcal{R}} = \frac{T}{\mu}(\Psi)$

$\therefore$ If chemically homog, $T = T(\Psi)$ too, but if $\mu \neq$ const. $T \propto \mu$ on such equipot. surfaces, and minor deviations from hydrostatic equilibrium $\Rightarrow$ changes in T from inertia terms

<u>Von Zeipel's Theorem</u> redux: radiative transport in a homog. hydrostatic ☆ w/ conservative rotation. Then radiative transport eqn is: $\vec{F} = -\frac{4ac}{3\kappa\rho} T^3 \nabla T$ (10)

For these circumstances, $T = T(\Psi)$; let $-\nabla\Psi \equiv \vec{g}_{eff}$

$$\text{so } \vec{F} = \frac{4ac}{3\kappa\rho} T^3 \frac{dT}{d\Psi} \vec{g}_{eff} = -k(\Psi)\, \vec{g}_{eff} \qquad (11)$$

$$\therefore \vec{\nabla}\cdot\vec{F} = -\frac{dk(\Psi)}{d\Psi}(\nabla\Psi)^2 - k(\Psi)\left(4\pi G\rho - \frac{1}{\varpi}\frac{d(\varpi^2\Omega^2)}{d\varpi}\right) = \epsilon\rho \qquad (12)$$

using (8), $\frac{\partial \Omega}{\partial z} = \frac{\partial \Omega}{\partial \varphi} = 0$ & $\nabla^2 \Phi = 4\pi G\rho$

$\exists$ a BIG PROBLEM here! Since homog, P, ρ, T const. on $\Psi = \text{const}$ surfaces. $\therefore \epsilon\rho$ & $k \cdot 4\pi G\rho$ are const, but

$$\frac{dk}{d\Psi}(\nabla\Psi)^2 \neq k\,\frac{1}{\varpi}\,\frac{d(\varpi^2\Omega^2)}{d\varpi}$$

so (12) can't be satisfied in general. Check for solid body rot'n: $\Omega = \text{const}$ so $\frac{1}{\varpi}\frac{d(\varpi^2\Omega^2)}{d\varpi} = 2\Omega^2 = \text{const}$.
BUT $(\nabla\psi)^2$ varies on equipot. surfaces, since $g_{eff,eq} < g_{eff,p}$
i.e. radiative transport & simple energy conservation can't be fulfilled simultaneously $\Leftrightarrow$ Van Zeipel's Thm.
So what actually happens? Vogt & Eddington showed rad. trans $\Rightarrow \exists$ regions in ★ that cool off & other regions that heat up $\Rightarrow$ bouyancy & meridional circulation.
This can also be driven by tidally or magnetically distorted ★s (as well as rotationally distorted).
What is this meridional circulation velocity?

Eqn of motion is: $$\frac{d\vec{v}}{dt} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi \qquad (13)$$
In inertial frame of ref: $$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \qquad (14)$$
$$\rho c_v\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T\right) + P\vec{\nabla}\cdot\vec{v} = \rho\epsilon - \vec{\nabla}\cdot\vec{F} \qquad (15)$$
Restrict ourselves to axisymmetric motions: $\vec{v} = \vec{v}_\varphi + \vec{u}$ (meridional)
$$\vec{v} = \Omega\varpi\,\hat{e}_\varphi + \vec{u} \qquad (16)$$
$\therefore$ the ϖ component of (13) can be written as
$$\frac{d\vec{u}}{dt} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi + \Omega^2\varpi\,\hat{e}_\varpi \qquad (17)$$
The $\hat{\varphi}$ component is cons. of ang. mom:
$$\frac{\partial}{\partial t}(\Omega\varpi^2) + \vec{u}\cdot\nabla(\Omega\varpi^2) = 0 \qquad (18)$$
Axial symmetry also $\Rightarrow$ in (14) & (15) we replace $\vec{v}$ by $\vec{u}$ $\therefore$ Have 7 relations between $\Omega, \vec{u}, P, \rho, T$ & ϕ if we include (10), $\nabla^2\phi = 4\pi G\rho$ & $P = \frac{R}{\mu}\rho T$.
In principle, can be solved if BCs & ICs known --- but too complex.
Make further assumptions: Steady-state: $\frac{\partial}{\partial t} = 0$ & slow. $|\vec{u}| \ll |\Omega\varpi|$. Then (14) & (15) become:

$$\nabla \cdot (\rho \vec{u}) = 0 \qquad (19)$$

$$\rho C_v \vec{u} \cdot \vec{\nabla} T + P \vec{\nabla} \cdot \vec{u} = \rho \epsilon - \vec{\nabla} \cdot \vec{F} \qquad (20)$$

Then (17) & (18) reduce to:

i.e.

$$\frac{1}{\rho} \vec{\nabla} P + \vec{\nabla} \phi = \Omega^2 \varpi \hat{e}_\varpi \qquad (21)$$

$$\vec{u} \cdot \vec{\nabla} (\Omega \varpi^2) = 0 \qquad (22)$$

Rad. trans. eqn, Laplace's eqn. & EoS are unaffected.
This system can be solved as follows:

Build a model satisfying (21) w/ a prescribed $\Omega(\varpi)$ distribution. Compute $\vec{u}$ from (19) & (20).

BUT not consistent w/ strict ☆ hydrodynamics since they must also comply w/ (22) which ⇒ steady meridional flow in an inviscid, non-magnetic ☆ must follow surfaces of const. ang. mom. / mass.

However, a steady circ. pattern must have closed streamlines & these 2 requirements <u>can't</u> be exactly accomodated simultaneously.

∴ Such "steady-state" sol'ns <u>must</u> distort the assumed rotational field on a timescale:

$$t_{circ} \simeq R/u_c \qquad w/ \quad u_c = \langle u \rangle \qquad (23)$$

i.e., ∃ feedback from meridional circulation on rotation which ⇒ new merid. circ. current — constant slow evolution.

For $\frac{\partial \Omega}{\partial z} = 0$ (pseudo-barytropic rot'n) we have (8) & (9)

Using (19) & (9) we rewrite (20) as:

$$\nabla \cdot \left[\rho \vec{u} \left(C_v T + \frac{P}{\rho} + \Psi \right) \right] = \rho \epsilon - \vec{\nabla} \cdot \vec{F} \qquad (24)$$

If the ☆ is also chemically homog. then $P = P(\Psi)$, $\rho = \rho(\Psi)$, $T = T(\Psi)$.

We integrate $\rho \vec{u} \left(C_v T + \Psi + \frac{P}{\rho} \right)$ over a level surface & use (19) to find:

$$\oint_{S_\Psi} \rho \vec{u} \left(C_v T + \frac{P}{\rho} + \Psi \right) \cdot d\vec{S} = \left(C_v T + \frac{P}{\rho} + \Psi \right) \oint_{S_\Psi} \rho \vec{u} \cdot d\vec{S} = 0 \qquad (25)$$

w/ S_ψ the level surface $\psi = \text{const}$ & total flux of matter across such a surface must vanish. Using (25) we can rewrite (24) in integral form w/ Gauss' Law:

$$l_\psi = \int_{V_\psi} \rho \epsilon \, dV = \oint_{S_\psi} \vec{F} \cdot d\vec{S} \qquad (26)$$

i.e., even though radiative $\rightleftharpoons$ doesn't hold @ every point, it does hold on average. $\Rightarrow$ meridional circulation in a pseudo-barytropic ★ transports no net energy over a level surface.

Simple case of perturbation theory to find circulation (Sweet) velocity — a chemically homog. ★ is compelled to rotate more rapidly. Use spherical coordinates. Then mechanical $\rightleftharpoons$ (22) becomes:

$$\frac{\partial P}{\partial r} = -\rho \frac{\partial \phi}{\partial r} + \rho r \Omega^2 \sin^2\theta \qquad (27)$$

$$\frac{\partial P}{\partial \theta} = -\rho \frac{\partial \phi}{\partial \theta} + \rho r^2 \Omega^2 \sin\theta \cos\theta \qquad (28)$$

Expand in terms of $\lambda = \frac{\Omega^2}{4\pi G \rho_c} = \frac{1}{2}\chi'' \qquad (29)$

ρ_c central density of unperturbed ★:

i.e.

$$\phi(r,\theta) = 4\pi G \rho_c k^2 [\phi_0(r) + \lambda \phi_1(r,\theta) + \lambda^2 \phi_2(r,\theta) + \dots] \qquad (30)$$

$$\rho(r,\theta) = \rho_c [\rho_0(r) + \lambda \rho_1(r,\theta) + \lambda^2 \rho_2(r,\theta) + \dots] \qquad (31)$$

$$T(r,\theta) = T_c [T_0(r) + \lambda T_1(r,\theta) + \lambda^2 T_2(r,\theta) + \dots] \qquad (32)$$

w/ $k^2 = 5RT_c / 8\pi\bar{\mu} G \rho_c \qquad (33)$

One can then measure r in units of k.

We'll just examine the 1st order terms. The 2nd are also useful, but messier & can be found using similar techniques.

It can then be shown:

$$\chi_1(r,\theta) \equiv \frac{\partial \phi_1}{\partial \theta} = r^2 f(r) \sin\theta \cos\theta \qquad (33)$$

& then that the radial fn. satisfies the nonhomog. eqn:

$$\frac{d^2 f}{dr^2} + \frac{6}{r}\frac{df}{dr} - \frac{1}{g_0}\frac{d\rho_0}{dr}(f-1) = 0 \qquad (34)$$

where $g_0 \equiv \frac{d\phi_0}{dr}$ is the grav. accel in the known unperturbed configuration. Once $f(r)$ is found $\frac{\partial \rho_1}{\partial \theta}$ & $\frac{\partial T_1}{\partial \theta}$ can also be found, using EoS, (28), & above procedure

Next step involves finding u_r & u_θ in terms of known fns.

$$u_{r,\theta} = \lambda u_{1r,1\theta}(r,\theta) + \lambda^2 u_{2r,2\theta}(r,\theta) \qquad (35)$$

& $$F(r,\theta) = F_0(r) + \lambda F_1(r,\theta) + \lambda^2 F_2(r,\theta) \qquad (36)$$

Return to (20) which can be written as:

$$\rho T \vec{u}\cdot\vec{\nabla} S = \rho \epsilon - \vec{\nabla}\cdot\vec{F} \qquad (37)$$

w/ $S = c_v \log \frac{P}{\rho^\gamma} + \text{const}$ is entropy/mass.

Then, outside the nuclear burning region, to 1st order in λ:

$$\rho_0 T_0 \frac{dS_0}{dr} u_{1r} = -\vec{\nabla}\cdot\vec{F}_1 \qquad (38)$$

(38) $\Rightarrow$ rot. causes a net influx of energy @ each point ($-\vec{\nabla}\cdot\vec{F}_1$) & rate of circulation must adjust $\ni$ this energy is convected away. Differentiate (38) & (10) w.r.t. θ to get $\frac{\partial u_{1r}}{\partial \theta} = Q(r,\theta)$ ~ a known fn.

$$\therefore u_{1r}(r,\theta) = \int_0^\theta Q(r,\theta')d\theta' + Q_1(r) \qquad (39)$$

w/ Q_1 found from the condition that the total flux of matter over any spherical surface $r=$const. must vanish. This

$$\Rightarrow u_{1r}(r,\theta) = \int_0^\theta Q(r,\theta')d\theta' - \frac{1}{2}\int_0^\pi \sin\theta_1 \int_0^{\theta_1} Q(r,\theta)\,d\theta\, d\theta_1 \qquad (40)$$

The 2nd order term can be shown to lead to an overall expression:

$$u_r(r,\theta) = \frac{LR^2}{GM^2}\left\{\lambda\alpha_0(r)P_2(\theta) + \lambda^2\left[\alpha_2(r)P_2(\theta) + \alpha_4(r)P_4(\theta)\right] + \ldots\right\} \quad (41)$$

Here the α's are fn's depending on the initial spherical model. Then $u_\theta(r,\theta)$ is obtained from eqn. of continuity.

A specific case: Cowling model — Point-source of ϵ w/ const. μ & Ω. Up to 2nd order: (Maheswaran, M. MNRAS 140, 93 (1968))

$$u_r(r,\theta) = \frac{L/L_\odot\,(R/R_\odot)^2}{(M/M_\odot)^2}\,Y(r,\theta) \quad (42)$$

where

$$Y(r,\theta) = \frac{\left\{\lambda\beta_0(r)P_2(\theta) + \lambda^2\frac{\bar{\rho}}{\rho_0}\left[\beta_2(r)P_2(\theta) + \beta_4(r)P_4(\theta)\right]\right\}}{n - 3/2} \quad (43)$$

with $n \equiv \frac{d\log\rho_0}{d\log T_0}$, the effective polytropic index. (44)

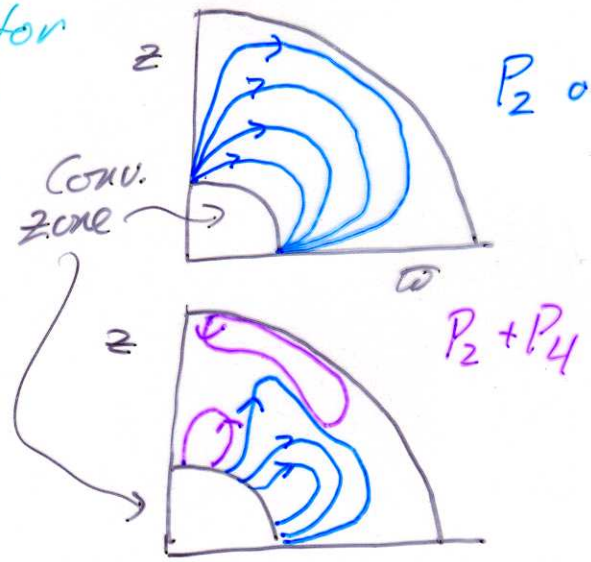
For this Cowling model, $\alpha = \frac{\Omega^2 R^3}{GM} = 1.11\times10^2\,\lambda$ (45)

pert. force / grav @ equator

Key results from numerical sol'n:

(i) Circulation pattern is in 2 regions w/ motions in opposite senses.

(ii) $|\vec{u}|$ is very small over nearly all of radiative zone. Exception is near convective zone where $n \to 3/2$ & near envelope, where $\rho_0 \to 0$.



But differential rotation will usually eliminate the circulation reversal seen in rigid rotators (Clement).

Eqns (42)–(45) allow an estimate of the typical circulation speed in the radiative zone:

$$u_c \simeq \frac{LR^2}{GM^2}\alpha \quad \therefore \text{ Global circulation time is:}$$

$$\therefore t_{Edd\text{-}Sweet} \simeq \frac{R}{u_c} = \frac{GM^2}{LR}\frac{1}{\alpha} = t_{K\text{-}H}\frac{1}{\alpha} > 10^{15}\,s \text{ for most } \star \quad (46)$$

N.B. Near surface $\bar{\rho}/\rho_0 \nearrow$ & local timescale is

$$t_{circ} \simeq \frac{GM^2}{LR}\frac{\rho_0}{\bar{\rho}}\frac{1}{\alpha^2} \qquad (47)$$

As $\bar{\rho}/\rho_0 \nearrow$ the assumption $|\vec{u}| \ll |\Omega \varpi|$ is violated & (47) is invalid.

Other timescales relevant to angular momentum distributions in stars deserve brief mentions: (Perhaps more later in discussing evolution):

Conservation of ang. mom. can be written as:

$$\frac{d(\varpi^2\Omega)}{dt} = \varpi^2\frac{d\Omega}{dt} + \vec{v}\cdot\vec{\nabla}(\varpi^2\Omega) = 0 \qquad (48)$$

$\sim$ large-scale $\vec{v}$ in ★.

What modifies it?

(i) Atomic motion & molecular viscosity

(ii) Radiative viscosity

(iii) Turbulent viscosity – convective motions

(i) Simple viscosity coef is: $\eta = \rho \underset{\text{mfp}}{l} v_{th}$ (49)

For ions $\eta \sim 1$ in cgs ($g\,cm^{-1}s^{-1}$).

Must use Navier-Stokes eqns for viscous fluid.

Key component is: $\rho\frac{\partial\Omega}{\partial t} = \eta\nabla^2\Omega$ (50)

Like heat transfer, so: $\tau_{visc,ion} \simeq \frac{d^2\rho}{\eta}$ (51)

w/ d the characteristic length, $\sim \frac{1}{\Omega}\left(\frac{d\Omega}{dr}\right)^{-1}$

Take $d \sim 10^{11}\,cm \sim R$, $\rho \sim 1\,g\,cm^{-3}$, $\eta \sim 1\,g\,cm^{-1}s^{-1}$

$\Rightarrow \tau_{visc} \sim 10^{22}\,s \gg \tau_★$ & TOO SLOW.

(ii) For photons $\eta_{rad} \simeq \rho_{rad}\, l_\gamma c$ (52)

$l_\gamma \simeq \frac{1}{\kappa\rho}$, $\rho_{rad} = \frac{aT^4}{c^2}$ $\therefore$ Use in (51) to get

$$\tau_{visc,rad} \simeq \frac{d^2\rho^2 c\kappa}{aT^4} \qquad (53)$$

$d = 10^{11}$, $\rho = 1$, $\kappa = 1\,cm^2g^{-1}$, $T = 10^7\,K \Rightarrow \tau_{visc,rad} \simeq 10^{18}\,s$, also too long.

∴ a ☆ is effectively a frictionless fluid, & if stable then only slow meridional circulation dominates.

iii) CONVECTIVE regions have: $\eta_{turb} \approx \rho l_{mix} v_{turb}$ (54)

For ⊙'s envelope, $v_{turb} \sim 0.01 c_s$, $l_{mix} \approx H_p \approx 10^8$ cm, $\rho \approx 10^{-4}$ g cm^{-3} & $c_s \approx 2\times10^6$ cm s^{-1} as $T \sim 3\times10^4$ K

$$\Rightarrow \eta_{turb} \simeq 2\times10^8 \text{ g cm}^{-1}\text{ s}^{-1} \qquad (55)$$

$$\therefore \tau_{turb} \approx \frac{d^2\rho}{\eta_{turb}} \approx \frac{(2R_{conv}^2)\rho_{conv}}{2\times10^8} \approx \frac{4\times10^{20}\cdot10^{-4}}{2\times10^8} \approx 2\times10^8 \text{ s} \sim 7 \text{ yr!}$$

In reality, while viscosity is actually a tensor & anisotropic, & while it is driven towards solid body rotation in convective zones, some differential rotation is allowed (cf ⊙'s surface).

Without actually solving full 2-D (axisymmetric) or 3-D equations one can get good models for moderately rotating ☆s with the Quasi-Spherical approximation. This can be plugged into stellar evol. eqns. Pioneered by Kippenhahn & Thomas; nice extensions done by Endal & Sofia.

Consider a chemically homogeneous, quasi-spherical ☆ in uniform rotation. Then P, ρ, Ψ & T have constant values on the set of surfaces denoted by $\Psi(a)$ with a the mean radius, & then the effective gravity is:

$$g = |g| = \frac{d\Psi}{dn} \qquad (56)$$

with n the normal to the level surface.

Let $\mathcal{A}_a$ be the area of the surface $\mathcal{S}_a$ & define means:

$$\langle g \rangle = \frac{1}{\mathcal{A}_a}\oint_{\mathcal{S}_a} g\, dS \qquad \langle g^{-1} \rangle = \frac{1}{\mathcal{A}_a}\oint_{\mathcal{S}_a} g^{-1}\, dS \qquad (57)$$