Basic Equations

Mass conservation: Eulerian form: fix r
\[ \dot{\rho} = \dot{\rho}(r, t) \]
\[ \frac{d m}{d t} = 4\pi r^2 \rho \, dr - 4\pi r^2 \rho \, \dot{v} \, dt \]
\[ \text{shell volume} \]
\[ \text{(mass flux into or out of region)} \]

Typically we assume HYDROSTATIC EQUILIBRIUM, i.e. \[ \frac{d m}{d t} = 0, \] so:
\[ \frac{d m}{d r} = \frac{d m}{d r} = 4\pi r^2 \rho \] (1) or \[ \frac{d m}{d r} = 4\pi r^2 \rho \]

Just considering time variations:
\[ \frac{d m}{d t} = -4\pi r^2 \rho \, \dot{v} \] (2)

The general equation of mass conservation is:
\[ \frac{d \rho}{d t} = 0 = \frac{d \rho}{d t} + \nabla \cdot (\rho \nabla) \] (3)

In spherical coords. we obtain this from (1) + (2):
\[ \frac{d}{d t} \left( \frac{d m}{d r} \right) = \frac{d}{d t} \left( 4\pi r^2 \rho \right) = \frac{d}{d r} \left( \frac{d m}{d t} \right) = \frac{d}{d r} (-4\pi r^2 \rho \, \dot{v}) \]
\[ \text{or} \quad 4\pi r^2 \frac{d \rho}{d t} = -4\pi r^2 \rho \, \dot{v} \]

The Lagrangian form is often more useful where \( \rho_r \) (or \( \rho \)) is the independent variable, not \( r \). This is particularly true for spherical symmetry, i.e. non-rotating, non-magnetic stars.
Lagrangian: \( p = p(m,t) \) & \( r = r(m,t) \)

An advantage for evolutionary models as \( \dot{M} = 0 \) but \( \dot{R} \) can be large.

\[ 0 \leq m \leq M \Rightarrow 0 \leq r < R \]

Derivatives are related by:

\[ \frac{d}{dm} = \frac{1}{r} \cdot \frac{dr}{dm} \quad (4) \]

and

\[ \frac{d}{dt} \bigg|_m = \frac{1}{r} \left( \frac{dr}{dt} \bigg|_m + \frac{dr}{dt} \right) \quad (5) \]

Use (4) on \( m \Rightarrow \frac{dm}{dm} = \frac{dm}{dr} \frac{dr}{dm} \Rightarrow 1 = 4\pi r^2 \frac{dr}{dm} \]

or

\[ \frac{dr}{dm} = \frac{1}{4\pi r^2 p} \quad (6) \]

\[ \therefore \text{In general the operators transform as:} \]

\[ \frac{1}{m} = \frac{1}{4\pi r^2 p} \frac{1}{r^2} \]

Eqn (5) is equivalent to the substantiative or advective time derivative and conservation laws are much simpler written in terms of them.
Equations of State

1) Ideal gas (mixture)

\[ P_g = \sum \frac{n_i}{m_i} \rho i k T = \rho \frac{k T}{M} \]

and

\[ \mu^{-1} = \sum \hat{n}_i \left( \frac{m_i}{m} \right) \]

or

\[ \mu^{-1} = 2X + \frac{3}{4} Y + \frac{1}{2} Z \]

Recall polytropic relations:

\[ PV^{\Gamma_1} = \text{const.} \]

\[ P^{1-\Gamma_2} T^{\Gamma_2} = \text{const.} \]

\[ T V^{\Gamma_3-1} = \text{const.} \]

\[ \Gamma_1 \text{ for dynamical stability} \]

\[ \Gamma_2 \text{ for convective stability} \]

\[ \Gamma_3 \text{ for pulsational stability} \]

For an adiabatic process in ideal gases:

\[ \Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3 \text{ if monatomic} \]

2) Radiation pressure:

\[ P_r = \frac{1}{3} a T^4 = \frac{4 \pi}{3} \frac{k}{c^3} T^4 \]

\[ a = \frac{8 \pi^5 h^4}{15 c^3 h^3} = 7.565 \times 10^{-15} \text{ erg cm}^{-3} K^{-4} \]

\[ j \sigma = \frac{ac}{4} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} K^{-4} s^{-1} \]

Radiation energy density:

\[ U_r = a T^4 \]

For most stars, \( P = P_g + P_r \) & we often define:

\[ \beta = \frac{P_g}{P} \quad \text{or} \quad 1 - \beta = \frac{P_r}{P} \]

and if \( \beta \) is constant:

\[ P = \frac{\rho k T}{\beta} = \beta \mu m_H \]
3) **Degeneracy Pressure**

In an ideal gas the chance of being in any particular momentum state is negligible, but as $T$ & $N$ all fermions tend to drop into lowest energy states & if particle energy $\leq \mu f(p) \to 0$

Assume that the electron gas is completely degenerate, with $N$ e's in volume $V \Rightarrow$ each state has a label, but Pauli exclusion principle prevents all from hitting lowest energy level. If the volume $V$ is big enough, # of quantum states is:

$$f(p) dp \leq V \frac{8\pi p^2 dp}{h^3} \sim 2 \text{ states per phase space of vol. } h^3$$

If completely degenerate all lowest states filled $\Rightarrow$

$$f(p) = V \frac{8\pi p^2}{h^3}$$

If $\exists$ only $N$ e's then their momenta are bounded so that:

$$N = V \frac{8\pi}{h^3} \int_0^{P_F} p^2 dp = V \frac{8\pi}{3h^3} P_F^3$$

$$\Rightarrow P_F = \left(\frac{3h^3 N_0}{8\pi}\right)^{1/3} \text{ with } N_0 = N/V \text{ the electron density}$$

**What is the pressure?**

$P$ is the mean rate of transfer of momentum across an ideal surface of unit area in the gas: if $v_P$ corresponds to $p$, then:

$$PV = \frac{1}{3} \int_0^{P_F} f(p) p v_P dp$$

is a general relation for $x, y, z$ components.
In this case, use $E$ as the kinetic energy of an electron w/ momentum $p$, so $V_\rho = \frac{\partial E}{\partial p}$ as canonical variable.

\[ P = \frac{8\pi}{3h^3} \int_0^{P_F} p^3 \frac{\partial E}{\partial p} \, dp \]

Let $U$ be the internal energy of gas due to KE of individual e's, so:

\[ U = \int_0^{P_F} f(p)E dp \]

In general, and:

\[ U = V \frac{8\pi}{h^3} \int_0^{P_F} E p^2 \, dp \]

for complete degeneracy @ $T=0$.

In general $f(p) = \left[ e^{E - \mu}/h^3 + 1 \right]^{-1}$, the Fermi-Dirac distribution, w/ $\mu$ the fermion chemical potential, must be used.

Two limiting cases:

a) Non-relativistic: \( \vec{p} = m \vec{v}, \ E = \frac{p^2}{2m}, \ \frac{\partial E}{\partial p} = \frac{p}{m} \)

\[ P = \frac{8\pi}{3h^3} \int_0^{P_F} \frac{p^4}{m} \, dp = \frac{8\pi}{15h^3m} P_F^5 \]

\[ U = V \frac{8\pi}{h^3} \int_0^{P_F} \frac{p^4}{2m} \, dp \]

or \( P = \frac{2}{3} U \), with \( P_F = \left( \frac{3h^3 n_e}{8\pi} \right)^{1/3} \), we have

\[ P = \frac{4}{5m} \left( \frac{3h^3}{8\pi} \right)^{2/3} n_e^{5/3} \]

\[ \nu_F = \frac{P_F^2}{2m} \Rightarrow U = \frac{3}{5} n_e \nu_F \]

\[ P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m} n_e^{5/3} \]
b) Relativistic, via general case:

\[ E_P = \gamma mc^2 = mc^2 \left\{ (1 + \frac{P^2}{m^2c^2})^{1/2} - 1 \right\} \]

\[ \Rightarrow \frac{dE}{dP} = \frac{P}{m} \left( 1 + \frac{P^2}{m^2c^2} \right)^{-1/2} \]

Then,

\[ P = \frac{8\pi}{3m} \int \frac{P_f^4 \, dp}{\sqrt{\left(1 + \frac{P^2}{m^2c^2}\right)}} \]

Let \( \Theta \) be defined as:

\[ \sinh \Theta = \frac{P}{mc} \quad (\sinh \Theta_f = \frac{P_{E_f}}{mc}) \]

\[ P = \frac{8\pi^2 m^4 c^5}{3h^3} \int_0^{P_f} \frac{p^4 \, dp}{\sinh^4 \Theta} \]  

\[ = \frac{8\pi^2 m^4 c^5}{3h^3} \left[ \frac{\sinh^3 \Theta \cosh \Theta}{4} - \frac{3 \sinh 2 \Theta}{16} + \frac{3 \Theta}{8} \right] \]

Letting \( \chi = \frac{P_{E_f}}{mc} \) we get:

\[ P = \frac{8\pi^2 m^4 c^5}{3h^3} f(x) = 6.01 \times 10^{-22} \times f(x) \]

where

\[ f(x) = x (2x^2 - 3) (x^2 + 1)^{1/2} + 3 \sinh^{-1} x \]

and

\[ N = \frac{8\pi^2 m^3 c^2}{3h^3} x^3 \Rightarrow \]

\[ U = \frac{8\pi^2 m^4 c^5}{3h^3} g(x) \]

where

\[ g(x) = 8x^3 \left[ (x^2 + 1)^{1/2} - 1 \right] - f(x) \]

In the limits:

\[ f(x) \rightarrow \frac{8}{5} x^5 - \frac{4}{7} x^7 + \frac{1}{3} x^9 - \frac{5}{22} x^{11} + \ldots \quad (x \rightarrow 0) \]

\[ \rightarrow 2x^4 - 3x^2 + \ldots \quad (x \rightarrow \infty) \]

\[ g(x) \rightarrow \frac{12}{5} x^5 - \frac{3}{5} x^7 + \frac{1}{7} x^9 - \frac{15}{178} x^{11} + \ldots \quad (x \rightarrow 0) \]

\[ \rightarrow 6x^4 - 8x^3 + 7x^2 - \ldots \quad (x \rightarrow \infty) \]

The MR case is for \( x \rightarrow 0 \), already have. The extreme rel. is for \( x \rightarrow \infty \), then:

\[ P = \frac{1}{8} (\frac{3}{4})^{4/3} h c n^{4/3} \]

\[ U = 3P = \frac{3}{4} n E_F \]

Note that for neutrinos, \( E_\nu = pc \) & \( v_\nu = c \) so these hold exactly @ \( T=0 \).
Corrections for Finite Temperature

As long as $kT < \mu$ we can expand

$f(p) = \frac{\ln(1 + \exp [-(E - \mu)/kT])}{1 + \exp [(E - \mu)/kT]}$ in expressions for $U$ and $P$.

Results, to lowest order, are:

\[ P(T) = P(0) + \frac{\Pi^2}{8} \frac{n (kT)^2}{E_F} \]
\[ U(T) = U(0) + \frac{\Pi^2}{2} \frac{n (kT)^2}{E_F} \]
\[ \mu(T) = E_F - \frac{\Pi^2}{12} \frac{(kT)^2}{E_F} + I_0 \text{ (const)} \]
\[ C_V = \frac{3\Pi^2}{2} g_s \frac{k^2 T}{m c^2} \frac{\sqrt{1 + x^2}}{x} n \]

Only energy above Fermi level @ $E_F$ can be radiated away.

At very high temperatures pair production gets to be important:

- \( kT < \frac{2m c^2}{\lambda} \) then pairs dominate
- \( kT < \frac{2m c^2}{\lambda} \) get some pairs from the tail of the photon distribution \( \gamma + \gamma \rightleftharpoons e^- + e^+ \) is easiest. Use Fermi-Dirac funs:

\[ N^\pm(p) = \frac{1}{e^{(E_p - \mu^\pm)/kT} + 1} \]

Pair-prod. (or other) reactions are in \( \sum \)

\[ \sum \mu_{\text{LHS}} = \sum \mu_{\text{RHS}} \iff 2\mu_y = 0 = \mu_+ + \mu_- \]

\[ \therefore \mu = -\mu_- \] and

\[ \text{photons are massless Bosons} \quad (s = 1) \]
\[ n^-(p) = \frac{1}{e^{(E-m)/kT} + 1} \quad n^+(p) = \frac{1}{e^{(E+m)/kT} + 1} \]

In very late stages: \( \delta + \delta \rightarrow \nu_e + \bar{\nu}_e \)
\[ \nu_e + \bar{\nu}_e \rightarrow \mu^+ + \bar{\mu} \]
\[ \nu_e + 2\bar{\nu}_e \rightarrow \mu^+ + 2\bar{\mu} \]

and
\[ \nu_e^+ + \nu_e^- \rightarrow \mu^+ + \bar{\mu} \]

are important, as are
\[ e^+ + e^- \rightarrow \nu_e, \mu^+, \bar{\nu}_e, \bar{\mu} \]

and
\[ W \rightarrow \nu_e, \mu, \bar{\nu}_e, \bar{\mu}, \mu, \bar{\mu} \]

(Plasmon)

Corrections to ideal gas:
ion screening for \( n > 10^{12} \text{ g/cm}^3 \)

strong force for \( n > 10^{14} \text{ g/cm}^3 \)

\[ \log P \quad \log E \]

\[ E_{\text{Nuc}} \]

\[ \frac{P}{E} \]

Ideal

Repulsive Core

Attractive Nuc.