HOW TO BUILD A STELLAR MODEL

Using mass as the indep. var. the basic eqns of stellar structure are:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \qquad (1)$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \qquad (2)$$

$$\frac{\partial l}{\partial m} = \epsilon_n - \epsilon_\nu \underbrace{- c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}}_{\epsilon_g} \qquad (3)$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla \qquad (4)$$

rate @ which $j \to i$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) \qquad (5)$$

rate @ which $i \to k$

$i = 1, \ldots, I \quad \& \sum_i X_i = 1$ describes composition.

If written as time independent, & w/r as variable:

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad (1a)$$

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad (2a)$$

$$\frac{dl}{dr} = 4\pi r^2 \rho (\epsilon_n - \epsilon_\nu) \qquad (3a)$$

$$\frac{dT}{dr} = -\frac{GmT\rho}{r^2 P}\nabla \qquad (4a)$$

Now if we are in a radiative zone,

$$\nabla = \nabla_{rad} = \frac{3}{16\pi acG}\,\frac{\kappa \ell P}{m T^4} \qquad (4')$$

In the interior convective zone $\nabla = \nabla_{ad}$, otherwise more complicated things are needed, but

$\nabla \lessapprox \nabla_{ad}$ i.e. $\nabla_{ad} = \frac{P\delta}{T\rho c_p}$ w/ $\delta \equiv -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)$ from EoS

Eqn's (1-5) demand we know the properties of stellar matter & we must have

Constituitive Relations

$\rho = \rho(P, T, X_i)$ --- invert EoS (6)

$\Rightarrow \delta(P, T, X_i)$ --- needed for ∇_{ad}

$c_p = c_p(P, T, X_i)$ or $\Gamma_2(P, T, X_i)$ --- for ∇_{ad} (7)

then we have: $\nabla_{ad}(P, T, X_i)$

$\kappa = \kappa(P, T, X_i)$ (8)

$\epsilon_n = \epsilon_n(P, T, X_i)$ (9)

$\left.\begin{array}{l} \epsilon_\nu = \epsilon_\nu(P, T, X_i) \quad (10) \\ r_{jk} = r_{jk}(P, T, X_i) \quad (11) \end{array}\right\}$ less important for most phases

The full problem involves $4+I$ diff eqns for $4+I$ variables: $r, P, T, \ell, X_1 \ldots X_I$ with m & t the indep. vars. These are non-linear PDEs.

Assume $\dot{M} = 0$; then we need soln's for $0 \le m \le M$ & $t \ge t_0$.

Complete Equilibrium (mechanical + thermal) is usually valid. I.e. $\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}$ is very small

since: $\tau_{KH} \equiv \frac{|E_g|}{L} \simeq \frac{GM^2}{2RL} \simeq 1.6\times10^7 y$ for $\odot$

$\gg \tau_{ff} \simeq \left(\frac{R^3}{GM}\right)^{1/2} \simeq 27 min$ for $\odot$ $\left[\simeq (G\bar{\rho})^{-1/2}\right]$

Also: $\tau_n \equiv \frac{E_n}{L} \simeq 10^{11} y$ for $\odot$ (E_n ~ total energy avail.)

$\tau_n \gg \tau_{KH} \gg \tau_{ff}$ or τ_{hydro} on MS

As long as $\tau_n \gg \tau_{KH}$ then $-c_P\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}$ are negligible too.

Stellar models assume <u>slow</u> change in X_i so drop (5) and have <u>4 ODEs</u> (e.g. 1a-4a).

<u>Boundary Conditions</u> are split between central (simpler) and surface (more subtle).

Central BCs: @ $m=0$, $l=0$ & $r=0$ (12)

are easy, BUT $\nexists$ initial knowledge of P_c & T_c so these must be found by integrating inwards. However, $\exists$ general relations.

(1) $\Rightarrow$ $d(r^3) = \frac{3}{4\pi\rho}dm \Rightarrow r(m\simeq0) \simeq \left(\frac{3}{4\pi\rho_c}\right)^{1/3} m^{1/3}$ (13)

A similar expansion of (3) $\Rightarrow$

$l(m\simeq0) \simeq (\epsilon_n - \epsilon_\nu + \epsilon_g)_c\, m$ (14)

Use (2) & (13) to see $\left(\frac{\partial}{\partial t}=0\right)$

$$\frac{dP}{dm} \simeq -\frac{G}{4\pi}\left(\frac{4\pi\rho_c}{3}\right)^{4/3} m^{-1/3}$$

Integrate to $\Rightarrow$ $P - P_c = \frac{-3G}{8\pi}\left(\frac{4\pi}{3}\rho_c\right)^{4/3} m^{2/3}$ (15)

as $\frac{dP}{dr} = \frac{-Gm\rho}{r^2} \propto \frac{m}{r^2} \propto \frac{r^3}{r^2} \to 0$ as $m \to 0$.

We can integrate $\frac{\partial T}{\partial m}$ for both radiative & convective situations; this is so since for $P \to P_c$, $T \to T_c$, $K \to K_c$.

Use (13) & (14) so, **if radiative**:

$$\frac{dT}{dm} = \frac{-3}{64\pi^2 ac}\frac{K l}{r^4 T^3} = \frac{-3K_c}{64\pi^2 ac}\frac{(\epsilon_n - \epsilon_\nu + \epsilon_g)_c m}{\left(\frac{3}{4\pi\rho_c}\right)^{4/3} T^3 m^{4/3}}$$

or $$T^3 dT = \frac{-3K_c(\epsilon_n - \epsilon_\nu + \epsilon_g)_c (4\pi)^{4/3}}{64\pi^2 3^{4/3} ac}\rho_c^{4/3}\frac{dm}{m^{1/3}}$$

Integrate: $$T^4 - T_c^4 = -\frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3} K_c(\epsilon_n - \epsilon_\nu + \epsilon_g)\rho_c^{4/3} m^{2/3} \quad (16$$

But, if the central region is **convective**:

$$\frac{dT}{dm} = -\frac{T}{P}\frac{Gm}{4\pi r^4}\nabla_{ad,c} = \frac{-GT}{4\pi P_c}\nabla_{ad,c}\left(\frac{3}{4\pi\rho_c}\right)^{-4/3} m^{-1/3}$$

which integrates to: $$\ln T - \ln T_c = -\left(\frac{\pi}{6}\right)^{1/3} G\frac{\nabla_{ad,c}\rho_c^{4/3}}{P_c} m^{2/3} \quad (16$$

Surface BCs: Simplest choice:

@ $m \to M$: $P \to 0$ & $T \to 0$ (17)

this is true in that $\frac{P_s}{P_c} \lll 1$ & $\frac{T_s}{T_c} \lll 1$

But the real BCs, used to match model to obs

$$R = r(M) \quad \& \quad L = l(M) \quad (18)$$

Photospheric defn of R & T:

$$\tau = \frac{2}{3} = \int_R^\infty K\rho\, dr = \bar{K}\int_R^\infty \rho\, dr \quad (19)$$

In the atm, $g_0 = \frac{GM}{R^2} \simeq$ const. so

$$P(r=R) = \int_R^\infty g\rho\,dr = g_0\int_R^\infty \rho\,dr = \frac{GM}{R^2}\frac{2}{3}\frac{1}{K} \quad (20)$$

From the Stefan-Boltzman law:

$$T_{eff}(r=R) = \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4} \quad (21)$$

(20) & (21) clearly better than (17) BUT (19) assumes a diffusion approx — not really valid in atm.

Sophisticated BC's: Smooth transition from interior to atmosphere: choose m_F the fitting mass, $\lesssim M$, below the atm so interior eqns still valid, yet $l(m_F) \simeq L$.

Interior: M & X_i given

Then atmosphere theory says, for given M & X_i $\exists$ 2 parameter set of atm soln's. Let R & T_{eff} or R & L be these parameters.

Integrate downward until $m_F \rightarrow r_F^{ex}, P_F^{ex}, T_F^{ex}, l_F^{ex} = L$

Then go for outward integration of (1-4) using variations of P_c & T_c as 2 parameters until

$$r_F^{in} = r_F^{ex} \quad P_F^{in} = P_F^{ex} \quad T_F^{in} = T_P^{ex} \quad L_F^{in} = L_F^{ex} \quad (22)$$

Do lots of $\downarrow$ integrations for many $(R,L) \rightarrow q_F^{ex}(R,L)$ w/ L known, $R = R(r_F^{ex}; L)$. Invert so $P_F^{ex}(R(r_F^{ex}; L), L)$
$= \pi(r_F^{ex}; L)$

In principle one can always get solutions.

How Do Surface Conditions Affect Envelope Soln's?

Must consider radiative & convective envelopes separately

In the envelope, m varies very little, so P is a better variable

Radiative: $\frac{\partial T}{\partial P} = \frac{3}{64\pi\sigma G}\frac{\kappa l}{T^3 m}$ $\left(\frac{\partial T}{\partial m} \Big/ \frac{\partial P}{\partial m}\right)$ (23)

Assume: $\kappa = \kappa_0 P^a T^b$ w/ $a>0$ but $b<0$, typical

Then: $\frac{T^{3-b}}{P^a}\frac{\partial T}{\partial P} = \frac{3\kappa_0}{64\pi\sigma G}\frac{l}{m} \approx \text{const}$ (Far enough out $\ni l = L$, $m = M$)

$\therefore$ Integrate $T^{3-b}dT$ & $P^a dP$ to get:

$$T^{4-b} = B(P^{1+a} + C) \quad (24)$$

where $B = \frac{4-b}{1+a}\frac{3\kappa_0}{64\pi\sigma G}\frac{L}{M}$

§10.3 discusses various cases in detail.

Basically, for Kramer's opacity $a=1$, $b=-4.5$, (24) $\Rightarrow$

$$T^{8.5} = B(P^2 + C) \quad \& \quad \nabla = \frac{d\ln T}{d\ln P} = 0.235\frac{BP^2}{T^{8.5}} \quad (25)$$

If $C=0 \Rightarrow \frac{T^{8.5}}{BP^2} = 1$ or $\nabla = 0.235 < \nabla_{ad} = \frac{2}{5}$ (from $\Gamma = \frac{5}{3}$) $\nabla_{ad} = \frac{\Gamma - 1}{\Gamma}$

Consistent w/ radiative transfer and $T \to 0$ & $P \to 0$ so approaches zero BC.

$C>0 \Rightarrow \frac{T^{8.5}}{BP^2} > 1$ & $\nabla < \frac{2}{8.5}$. If $P^2 \ll C$, get $T^{8.5} = BC = \text{const}$

so finite values of T are approached as $P \to 0$.

Finally, as $P \nearrow$ the value of C is unimportant & sol'n $\to$ the $C=0$ sol'n.

$\therefore$ Precise starting values of atm. don't influence the interior very much! If radiative env. ($T_{eff} > 9000K$)

$C<0 \Rightarrow \nabla > \frac{2}{8.5}$ & rises until $\nabla = \nabla_{ad} \Rightarrow$ Convection sets in. Note that near surface ionization drops ∇_{ad} to < 0.4.

Convective envelope: small errors in T_{eff} allow diff. depths of convection for cooler ☆s & results are more sensitive than for rad. env.

Lower T_{eff} ⇒ deeper convective layer. ∃ difficulties with superadiabatic convection.

Simplest Complete Solutions

Solution via Internal Matching — start @ one end, integrate to the other, miss the BC's, adjust params, try again! This actually doesn't work numerically, but can set up eqn's.

Schwarzschild Variables — dimensionless, w/ r indep.

$$x = \frac{r}{R}, \quad q = \frac{m}{M}, \quad f = \frac{l}{L}, \quad p = P\left(\frac{GM^2}{4\pi R^4}\right)^{-1}$$

$$t = T\left(\frac{m_H \mu G M}{k R}\right)^{-1} \qquad (26)$$

Assume that: $\bar{\kappa} = \kappa_0 \rho^n T^{-s}$ & $\epsilon = \epsilon_0 \rho^\lambda T^\nu$ (27)

Then 1a), 2a) & 3a) respectively ⇒

$$\frac{dq}{dx} = \frac{p x^2}{t} \qquad (28a)$$

$$\frac{df}{dx} = D(\lambda,\nu)\, x^2 p^{\lambda+1} t^{\nu-\lambda-1} \qquad (28b)$$

$$\frac{dp}{dx} = \frac{-qp}{t x^2} \qquad (28c)$$

Convective: 4a ⇒ $\frac{d\ln p}{d\ln t} = 2.5$ (28d)

Radiative ☆: ⇒ $\frac{-C(n,s) f p^{n+1}}{x^2 t^{n+s+4}}$ (28e)

where 2 eigenvalues must be determined:

$$\left.\begin{aligned} C(n,s) &= \frac{3}{4(4\pi)^{n+2}ac}\left(\frac{k}{m_H G}\right)^{s+4}\frac{K_0}{\mu^{s+4}}\,\frac{L R^{s-3n}}{M^{s-n+3}} \\ D(\lambda,\nu) &= \frac{1}{(4\pi)^{\lambda}}\left(\frac{m_H G}{k}\right)^{\nu}(\epsilon_0 \mu^{\nu})\,\frac{M^{\nu+\lambda+1}}{L R^{\nu+3\lambda}} \end{aligned}\right\} (29)$$

N.B.: ρ was eliminated in favor of P&T & ∃ problem @ R since $P \to 0$ & $T \rightleftarrows 0$. But Schwarzschild shows that, as $x \to 1$, $t \simeq \frac{n+1}{n+7.5}\frac{1-x}{x}$ and $p \doteq \text{Const}\, t^{\left(\frac{n+7.5}{n+1}\right)}$ if the envelope is radiative.

Special Cases:

Convective core: The energy is produced where there is an adiabatic gradient. ∴ (28b) is redundant & we only need to find $C(n,s)$, not $D(\nu,\lambda)$ – this is called a Cowling model.

Solve Cowling by finding q_{core} & x_{core} via condition $\frac{d(\ln p)}{d(\ln t)} < 2.5$ When no longer true, shifts to a radiative envelope, w/ $l = l_c = L$ or $f = 1$ $\forall\ x > x_{core}$.

Radiative core: this is N.G. & ∃ numerical difficulties w/ (28). Then, rewrite with $p = \frac{P}{P_c}$ & $t = \frac{T}{T_c}$. Solve for f & q homologously (will discuss later) & get a new set of dimensionless D.E.s to replace (28). These are O.K. @ $r \to 0$ since $p \to 1$ & $t \to 1$, but then ∃ problems near surface where both p & $t \to 0$ for same value of x.

Shooting method: integrate from both inside & outside w/ different methods & make model agree in the middle.

Problems: Outward integration sep. by $\Delta P_c, \Delta T_c$ diverge towards surface as (4) w/ (4') $\Rightarrow T^{-4}$ factor $\Rightarrow$ blow-up near surface. Inward integration sep. by $\Delta R, \Delta L$ diverge toward middle as

$$\frac{\partial P}{\partial m} \propto r^{-4}$$

Evolutionary Models

If $\exists$ changes on times $\ll t_{thermal}$, then small surface changes can correspond to large internal changes.

If not quite in $\Rightarrow$ work done in (by) contraction or expansion must be reflected through changing entropy.

Over longer time scales, μ changes as nuclear reactions change the the composition.

Evolutionary models are usually based on:

Henyey Relaxation Method

Start w/ estimated model (e.g. Schwarzschild type method), then produce better models by solving $\forall$ variables @ once in a matrix scheme.

Key to stability: let m be indep var:

$$\frac{\partial r}{\partial m} = (4\pi r^2 \rho)^{-1} \qquad (30a)$$

$$\frac{\partial l}{\partial m} = \epsilon(P, \rho, \mu) - T\frac{dS}{dt} \qquad (30b)$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \qquad (30c)$$

$$\frac{\partial \ln T}{\partial \ln P} = f(P, T, \mu) \quad \text{---- a general form} \tag{30d}$$

Divide into N-1 zones, 1^{st} pt @ center, N^{th} @ surface

This yields finite difference equations:

$$\frac{r_{i+1} - r_i}{m_{i+1} - m_i} = \left(4\pi r_{i+\frac{1}{2}}^2 \rho_{i+\frac{1}{2}}\right)^{-1} \tag{31a}$$

$$\frac{l_{i+1} - l_i}{m_{i+1} - m_i} = \epsilon_{i+\frac{1}{2}} - T_{i+\frac{1}{2}} \left.\frac{\partial S}{\partial t}\right|_{i+\frac{1}{2}} \tag{31b}$$

$$\frac{P_{i+1} - P_i}{m_{i+1} - m_i} = -\frac{G M_{i+\frac{1}{2}}}{4\pi r_{i+\frac{1}{2}}^4} \tag{31c}$$

$$\frac{T_{i+1} - T_i}{P_{i+1} - P_i} = \frac{T_{i+\frac{1}{2}}}{P_{i+\frac{1}{2}}} f\left(P_{i+\frac{1}{2}}, T_{i+\frac{1}{2}}, \rho_{i+\frac{1}{2}}\right) \tag{31d}$$

We know m_i but have 8 unknowns for $N-1$ sets of 4 eqns — but most are overlapped & in the outer zone ∃ $N+1^{th}$ point ⇒ $4N-4$ equations, $4N$ unknowns and 4 BCs.

e.g. $r_1 = l_1 = 0$ are 2 & r_N (or T_N) & l_N @ surface are other 2. Will specify the sol'n & enable match to a stellar atm.

Typically 500 zones & 2000 eqns in models ⇒ BIG matrix to invert, but if ∃ estimated sol'n then it is a SPARSE matrix & can be solved via small corrections in a Newton-Raphsen type scheme.

So: start w/ Schwarzschild model

use previous Henyey as input to next evolution step when μ & ϵ changes are incorporated.