Relativistic Cosmology

Cosmological Principle - homogeneity & isotropy on large enough scale.

⇒ Friedmann's Equation

Standard Model is the Hot Big Bang.

4 problems but a huge number of successes:

1. Hubble expansion ($<10^9$ yr)
2. ~Isotropic microwave background radiation (decoupling) ($<10^6$ yr)
3. Abundances of light elements ($<10^3$)
4. Number of quark & lepton families ($<10^3$)

Won't discuss physical cosmology in this course ... some in 602, 8410.

Olber's Paradox

If space Euclidean & infinite sky's brightness should = $\pi$. This is because stars block each other out, otherwise: $I \to \infty$.

Absorbing gas doesn't work a. it would heat up. Still a problem if non-Euclidean, open or closed as long as it is infinitely old.

If stars & universe have finite age brightness drops but full darkness allowed only if universe is expanding & Doppler shift of expansion work lowers $T_{sky}$. 
Cosmological Term

Vacuum field eqn: \[ G_{ab} = 0 \] (1)

was clearly correct to Einstein, but didn't fulfill
pure Mach's principle since Minkowski space is a soln
-and an empty space would have inertial properties.

He could generalize his standard field eqns:
\[ G_{ab} = 8\pi T_{ab} \] (2)

in order to produce a static universe, that seemed
to be most logical. "Perfect" Cosmological
Principle: no change of \( t \) or space [how, \( \Lambda \) is?]
(2) could not \( \Rightarrow \) "Steady State". Rather, this
requires: \[ G_{ab} - \Lambda g_{ab} = 8\pi T_{ab} \] (3)

\( \Lambda \) = cosmological constant

-Machian since flat space no longer a soln
But de Sitter found for a vacuum soln \( w, \Lambda \neq 0 \) so
Einstein concluded \( \Lambda \) was "the biggest mistake I ever made."
Since: \[ g_{ab} = 0 \] (3) is consistent with \( T_{ab} = 0 \)
\( \Lambda \) the Lagrangian is: \[ \mathcal{L} = -\frac{1}{2} (R - 2\Lambda) + L_{\text{Matter}} \] (4)

Homogeneous? - Superclusters, sheets,
voids, Great Attractor \( \Rightarrow \) is so only on
largest scales \( \sim 100 \) Mpc.

3 other solutions such as Bianchi models-
anisotropic with nine general classes...
Type I have "cigar"-like or "pancake-like"

\[ ds^2 = t^2 dx^2 + t^2 dy^2 + t^2 dz^2 - dt^2 \] (5)

\( w/ \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \)
Weyl's Postulate: Adopt GR as correct local physics & add privileged observers - those associated w/ smeared-out galactic motion. A sub-stratum that pervades space & galaxies are particles in this flow. Mathematically: The particles of the sub-stratum lie in space-time on a congruence of time-like geodesics diverging from a point in the finite of infinite past. Physically: the sub-stratum can be taken as perfect fluid.

Relativistic Cosmology demands:

i) Cosmological Principle
ii) Weyl's Postulate
iii) General Relativity

Have maximally sym. space & comoving coords in a space of constant curvature. We'd already shown that a 3-space of const. curvature has:

\[ ds^2 = \frac{dr^2}{1-Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(6)

The line element for relativistic cosmology is:

\[ ds^2 = R(t)^2 \left( \frac{dr^2}{1-Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) - dt^2 \]  

(7)

One makes the conformally flat version by mapping:

\[ r = \frac{T}{(1+\frac{1}{4}Kr^2)} \]  

(8)

so that (7) then becomes:

\[ ds^2 = R(t)^2 \frac{dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)}{(1+\frac{1}{4}Kr^2)^2} - dt^2 \]  

(9)
The standard form arises by rescaling $r$ and $K$, hence $R(t)$:

$$r \rightarrow r' = |K|^{1/2} r, \quad K = |K| k, \quad R \rightarrow R' = \frac{R}{|K|^{1/2}} (K+1) t^{1/2}.$$

Then $k = +1, -1, 0$, replaces $K$ in (7) and (9) ($K=0$)

**Recall** $k=+1 \Rightarrow$ spherical geometry, closed $k=0 \Rightarrow$ flat, $\left\{ w=0 \right\} \Rightarrow$ infinite in extent

$\left\{ k=-1 \right\} \Rightarrow$ open

E.g., $k=+1 \Rightarrow$ (7) is singular as $r \rightarrow 1$, so substitute:

$$\begin{align*}
  r &= \sin X, \\
  dr &= \cos X dX, \\
  (10) &\quad (1-c^2)^{1/2} dX
\end{align*}$$

$$\begin{align*}
  ds^2 &= R_0^2 \left[ dX^2 + \sin^2 X (d\theta^2 + \sin^2 \theta d\phi^2) \right] \\
  (11) &\quad R_0 = \text{radius of the universe}
\end{align*}$$

**Now embed this 3-surface in 4-D Euclidean space** with coordinates $\left( w, X, \theta, \phi \right)$:

$$w = R_0 \cos X, \quad X = R_0 \sin X \sin \theta \cos \phi, \quad \theta = R_0 \sin X \sin \phi, \quad \phi = R_0 \sin X \cos \phi.$$

One can then show (12) $\Rightarrow$

$$ds^2 = dw^2 + dx^2 + dy^2 + dz^2 \equiv (11) \quad \text{and also}$$

$$\begin{align*}
  w^2 + x^2 + y^2 + z^2 &= R_0^2 \\
  \left\{ R = 3 \right\} &\Rightarrow 3-D \text{ sphere in 4-DEuc.}
\end{align*}$$

$$\begin{align*}
  V &= \sum_{\omega=0}^{2\pi} \sum_{\theta=0}^{\pi} \sum_{\phi=0}^{2\pi} (R_0 \sin X d\theta)(R_0 \sin X d\phi) dX \\
  &= 2\pi R_0^3 (1-c^2)^{1/2} \quad \text{radius of the universe}
\end{align*}$$

This 3-space is the generalization of $S^2$ or 2-sphere to 3-D: $S^3$ in 3-space ... But not really embedded in 3-space - it is the totality of physical points outside it & a boundary for it. The space is bounded, closed or compact in topology

Whole $S^T$ is **cylindrical**: $R \times S^3$
Friedmann's Equation

Combining (3), (9), + Weyl's postulate:

\[ T_{ab} = (p + p) u_a u_b + p g_{ab} \]  
\[ \Rightarrow \]
\[ 3 \frac{R^2 + k}{R^2} - \Lambda = 8\pi p(t) \]  
\[ 2 \frac{R^2 + k}{R^2} - \Lambda = -8\pi p(t) \]  
\[ (\Lambda = \frac{k}{R^2}) \]

(17) has \( R \leftrightarrow \) EOM of motion
(16) has only \( R \leftrightarrow \) Integral of motion or ENERGY eqn.
Take \( \frac{\partial}{\partial t} \) (16) mult by \( \frac{1}{8\pi} \) and add to (17): \( -\frac{3\dot{R}}{8\pi R} \)

\[ \Rightarrow \frac{\dot{R}}{R} + 3p = -\frac{3}{8\pi} R \left( \frac{3R^2}{R^2} + \frac{3k}{R^2} - \Lambda \right) = -3p \frac{R}{R} \]

mult by \( R^3 \), this becomes:
\[ \frac{d}{dt}(\rho R^3) + p \frac{d}{dt}(R^3) = 0 \]  
\[ (18) \]

Consider particles in vol. \( V \): clearly sub-structure expansion \( \Rightarrow V \propto R^3(t) \) & if total mass-energy in vol is: \( E = pV \) then (18) is 1st Law of Thermo

or Cons. of Energy:
\[ dE + pdV = 0 \]  
\[ (19) \]

The same result comes from: \( T_{ab} ; b = 0 \)
& says pressure does work in the expansion.

In the current universe: \( p = p_m + p_\gamma + p_{\Lambda,\text{dark}} \approx 10^{-5} \) mbaryon
so \( p = 0 \) or a dust universe is OK now.

Then (17) integrates to:
\[ R(R^2 + k) - \frac{1}{3} \Lambda R^3 = C \]  
\[ (20) \]
Use (18) to see const. of integration is:
\[ C = \frac{\rho_0}{3} R_0^3 \]  
\[ (21) \]

essentially energy content of vol. \( V \), const. from (19).
As \( p = 0 \), (21) expresses cons. of mass \(-2\pi\)
mass in Euclidean spherical vol. Use(21) in (16), get:
\[
R^2 = \frac{C}{R} + \frac{\Lambda}{3} R^2 - k
\]  
(22)

Friedmann's Eqn. In a Newtonian approximation
(which must assume Hubble's law) \( R^2 \propto \) Kinetic Energy,
\( \propto \) Potential Energy, \( \Lambda > 0 \\Rightarrow \) Cosmic repulsion,
\( \Lambda < 0 \Rightarrow \) Cosmic attraction.

**Light Propagation** Assume Rel Cosmo
works like \( 6\pi R \), \( O \) receives light from
a distant galaxy \( P \). Use RW metric (7)
& homing spaces for slices so ungray take
\( r = 0 @ O \). Radial null geodesics says: \( ds^2 = d\theta d\phi = 0 \)
\[
- \quad \Rightarrow \quad \frac{dt}{R(t)} = \pm \frac{dr}{(1-kr^2)^{1/2}} 
\]
(23) receding \( + \), approaching \( - \)

Consider ray from \( P \) on world line \( r = r_i, @ t = t_i \),
\[
\therefore \int_{t_i}^{t_f} \frac{dt}{R(t)} = - \int_{r_i}^{r_f} \frac{dr}{(1-kr^2)^{1/2}} = f(r_i) = \int_{r_i}^{r_f} \frac{1}{\sqrt{1-kr^2}} \to k = +1
\]
\( \Rightarrow \) same \( f(r) \)
(24)

\( \therefore \int_{t_i}^{t_f+\Delta t} \frac{dt}{R(t)} = \int_{t_i}^{t_f} \frac{dt}{R(t)} = \sum \text{2 succesive light rays: } t_i, t_i+\Delta t, \to \text{ Same f(r)} \)
\[
\Rightarrow \int_{t_i}^{t_f} \frac{dt}{R(t)} = \int_{t_i}^{t_f} \frac{dt}{R(t)} - \int_{t_i+\Delta t}^{t_f} \frac{dt}{R(t)} + \int_{t_i}^{t_i+\Delta t} \frac{dt}{R(t)} = 0
\]

As \( R(t) \) doesn't change much over \( \Delta t \), or \( dt_0 \), take
it out of integral:
\[
\frac{dt_0}{R(t_0)} = \frac{dt}{R(t)} \]
(25)

But \( dt = ds \) on \( dt \) is proper time on substratum
world lines \& \( dt \), and \( dt_0 \) are proper times
as measured by source \& observer.
(25) says interval for $O$ is $R(h)/R(t_1)$ times that for $P$. In an expanding universe:

$$t_0 > t_1 \implies R(t_0) > R(t_1) \implies O \text{ experiences a RED SHIFT } z = \frac{1+z}{1+z} = \frac{R(t_0)}{R(t_1)} = (26)$$

**called a Doppler shift but not the same as SR Doppler shift**

If $P$ is near $O$ then $t_0 = t_1 + \Delta t$, so $(28) \implies$

$$1 + z = \frac{R(t_0)}{R(t_0 - \Delta t)} = \frac{R(t_0)}{R(t_0 - \Delta t)} \approx 1 + \frac{R(t_0)}{R(t_0 - \Delta t)} \frac{dt}{R(t_0)} \approx (27)$$

But also:

$$\int_{t_0}^{t_1} \frac{dt}{R(t)} = \int_{t_0}^{t_1} \frac{dt}{R(t)} = \frac{dt}{R(t)} = \frac{dt}{R(t)} \approx \frac{dt}{R(t)}$$

for small $\Delta t$, use $(24)$ to see

$$\int_{t_0}^{t_1} \frac{dt}{R(t)} = f(t) \approx R(t_1) \approx R(t_0)$$

Use $(27)$ to get what looks like Hubble's Law:

$$z \approx \frac{\dot{R(t_0)}}{R(t_0)} \approx (28)$$

**Cosmological Distance**

Absolute distance:

$$d_L = R(t) \int_0^t \frac{dr}{(1 - kr_1)^2} (29)$$

from $dt = d\theta = d\phi = 0$ but it is useless in practice.

An angular diameter distance, but the most useful is luminosity distance:

$$d_L^2 = \frac{L}{4\pi I(1+z)^2}$$

using luminosity and measured intensity $I$.

Now use $(27)$ & note light spreads from $P$ at $t_0$ to us at $O$ at $t_0$. 

Light will have spread over a sphere w/ center at $P_0$ ($t=t_0, r=r_0$), passing through $Q_0(t=t_0, r=0)$.

Surface area for this sphere is:

$$ds^2 = [R(t_0)]^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

So, as usual line element, the surface area is: $4\pi R^2(t_0) d\theta$

$$L = \frac{1}{4\pi R(t_0)^2}$$

Using (30), we see

$$d_L = r/ R(t_0)$$

(31)

Now define Hubble parameter: $H(t) = \frac{\dot{R}(t)}{R(t)}$ (32)

Then (28) + (32) + (34) $\Rightarrow Z = H(t_0)d_L$

(33)

Hubble's Law - only approx. in Rel. Cosmo.

$H(t_0) = H_0$ and $T_0 = \frac{1}{H_0} \approx 10^{10}$ yr.

Now define the deceleration parameter, $q$:

$$q(t) = -\frac{\ddot{R}^2}{\dot{R}^2}$$

(34)

Note as $R > 0 \& \ddot{R}^2 > 0 \Rightarrow \ddot{R} > 0 \Rightarrow \ddot{p} > 0 \Rightarrow q > 0 \Rightarrow$

slowing of expansion of the Universe. Direct measurements still very uncertain, but

$H_0 \approx 75 \pm 15$ km/s/Mpc$^{-1}$

and $q_0 = 0.5 \pm 1$ come from observations.

If we now take the 2nd order term in (27) we find:

$$d_L = z \frac{H_0}{c} \left[ 1 - \frac{1}{2}(1 + q_0)z + \cdots \right]$$

(35)

Differentiating Friedmann's eqm (22) $\Rightarrow$

$$2\ddot{R}^2 = -\frac{\dot{R}^2}{R^2} + \frac{2}{3} \Lambda R^2.$$  Multiply $\frac{\dot{R}}{2\dot{R}^2}$, use (34), (33), (21) to see

$$q = \frac{4\pi \rho - \frac{1}{3}\Lambda}{H^2}$$

($\Lambda = 0$: $q = \frac{4\pi \rho}{3H^2}$)

(36)
Cosmological Models

Want to solve Friedmann eqns:

$$\dot{R}^2 = \frac{3}{R} + \frac{1}{2} \Lambda R^2 - k$$  \hspace{1cm} (37)

Subject to: \( C > 0, \ -\infty < \Lambda < +\infty, \ k = -1, 0, 1 \) \hspace{1cm} (38)

Can solve via elliptic functions, but specialize to either \( k = 0 \) or \( \Lambda = 0 \)

Flat Space Solutions: \( k = 0 \), so:

$$\dot{R}^2 = \frac{C}{R} + \frac{1}{2} \Lambda R^2$$  \hspace{1cm} (39)

Assume \( \Lambda > 0 \) & let \( u = \frac{2 \Lambda}{3C} R^3, \ \ddot{u} = \frac{2 \Lambda}{3C} \dot{R}^2 \)

Plug into (39): \( \dot{u}^2 = \frac{4 \Lambda}{C} R^3 + \frac{4 \Lambda^3}{3C} \dot{R}^4 = 3 \Lambda (2u + u^3) \) \hspace{1cm} (40)

Assume \( + \text{ree } \):

\( \dot{u} = (3 \Lambda)^{1/2} (2u + u^3)^{1/2} \) \hspace{1cm} integrate by parts

Assume Big Bang I.C.: \( R = 0 @ t = 0 \Rightarrow \dot{u} = 0 \) so

$$\int_0^u \frac{du}{(2u + u^3)^{1/2}} = \int_0^t (3 \Lambda)^{1/2} dt = (3 \Lambda)^{1/2} t$$

Complete square in \( u \) int., let \( v = u + 1, \ \cosh w = v \):

$$\int_0^u \frac{du}{(2u + u^3)^{1/2}} = \int v \frac{dv}{(v^2 + 1)^{1/2}} = \int \frac{\sinh w dw}{(\cosh w - 1)^{1/2}} = \int_0^w dw = w$$

\( w = \text{arccosh}(v) \)

In terms of \( R \):

$$R^3 = \frac{3C}{2\Lambda} [\cosh(3\Lambda)^{1/2} t - 1]$$  \hspace{1cm} (41)

Assume \( \Lambda < 0 \), let \( u = -\frac{2 \Lambda}{3C} R^3 \) & find

$$R^3 = \frac{3C}{2(-\Lambda)} \left[ 1 - \cos(3\Lambda)^{1/2} t \right]^{1/2}$$  \hspace{1cm} (42)

For \( \Lambda = 0 \) \( t \text{th} \) \( \Rightarrow R^{1/2} dR = C^{1/2} dt \) and get

Einstein-de Sitter model: \( R = (\frac{3}{2} C)^{1/3} t^{2/3} \)  \hspace{1cm} (43)
One can then find $H(t)$ and $q(t)$ from (41), (42), and (43).

\[ H(t) = \frac{\dot{a}}{a} = \frac{\dot{R}}{R} = \frac{\dot{r}}{r} = \frac{1}{r} \]

\[ q(t) = -\frac{\dot{r}^2}{r^2} = \frac{1}{r^2} \]

Now, note that in the early stages of Big Bang, $R$ is small so $\frac{\dot{R}}{R} \gg \frac{1}{2} \dot{R}^2 R$ in (37).

\[ \dot{r}^2 \propto \frac{1}{r^2} \Rightarrow R \propto \left( \frac{t}{\sqrt{c}} \right)^{1/2} \]

so ALL Big Bang Models are $\Lambda = 0$ models @ small $t$.

Write (39) as: $R^2 = F(R)$, $w/F(R) = \frac{\dot{r}}{r}$ + $\frac{1}{2} R^2$

If $\Lambda < 0$, $F(R) = 0$ @ $R = R_m = \left( \frac{3c^4}{4\Lambda} \right)^{1/2}$

so $\dot{R} = 0$ @ $R_m$ --- a local minimum.

If $\Lambda > 0$, $\dot{R}$ grows w/o bound. Specifically, if $\Lambda > 0$, in large $t$, $\frac{1}{2} \dot{R}^2 \gg \frac{1}{2} \frac{\dot{R}}{R}$ can integrate:

\[ R \propto \exp \left[ (\frac{4\Lambda}{3c^4})^{1/2} t \right] \]

Consider Models w/ $\Lambda = 0$:

\[ \text{Case: } k = +1, \text{ let } \frac{\dot{r}^2}{c^2} = \frac{\dot{R}^2}{R^2} \frac{1}{k} \quad (48) \]

\[ \dot{r}^2 = \frac{\dot{R}^2}{c^2} = \frac{4\dot{R}^2}{4c^2 R^2} = \frac{4\dot{R}^2}{4c^2 R^2} (\dot{R}^2 - 1) = \frac{4\dot{R}^2}{4c^2 R^2} (\dot{R}^2 - 1) \]

Take $+ve \sqrt{}$; & BB L.C.'s $\Rightarrow 2 \int_0^t \dot{u}^2 \frac{da}{a} = \frac{1}{2} \int_0^t \dot{R}^2 \frac{da}{a} = \frac{t}{2}$

Let $u = \sin(\theta) \quad \Rightarrow 2 \int_0^t \dot{u}^2 \frac{da}{a} = 2 \int_0^t \dot{u}^2 \cos^2(\theta) \frac{da}{a} = 2 \int_0^t \dot{u}^2 \cos^2(\theta) \frac{da}{a} = 2 \int_0^t \dot{u}^2 \cos^2(\theta) \frac{da}{a} = \frac{t}{2}$

\[ C \left[ \sin^{-1}(\frac{1}{k}) \right]^2 - (\frac{1}{k})^2 (1 - Rc)^{1/2} \frac{1}{(1 - Rc)^{1/2}} = \frac{t}{2} \quad (49) \]

\[ \text{Case: } k = -1 \Rightarrow C \left[ \frac{1}{k} \right]^{1/2} (1 + Rc)^{1/2} - \sinh^{-1}(\frac{1}{k})^{1/2} = \frac{t}{2} \quad (50) \]

\[ \text{Case: } k = 0 \Rightarrow \text{Einstein-de Sitter again.} \]

\[ \text{NB, } k = +1: \quad H(t) = C^{-1} (\frac{R}{c})^{1/2} (1 - Rc)^{1/2} \quad (51) \]

\[ q = \frac{1}{2} (1 - Rc)^{-1} \quad (52) \quad \text{w/ } R \text{ given implicitly by } (49) \]
Classify Friedmann Models

<table>
<thead>
<tr>
<th>$\Lambda &gt; 0$</th>
<th>$\Lambda = 0$</th>
<th>$\Lambda &lt; 0$</th>
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<tbody>
<tr>
<td>$k = -1$</td>
<td>$k = 0$</td>
<td>$k = +1$</td>
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Fig. 23.1: Classification of Friedmann models.

1st Model: Einstein: no expansion (1916)
2nd De Sitter [next page] (1917)
3rd Eddington-Lemaître (1925) - Einstein's world is unstable, so why old, so Einstein, then expands, but at increasing pace $\Rightarrow \rho < 0$
But galaxies should stop forming
4th Lemaître (1935) Nucleogenesis demanded high $\rho$, $\phi$, $T$ early on $\Rightarrow$ Hot Big Bang. This had
(i) expansion from singularity + elements
(ii) slow expansion to $\Rightarrow$ condensation to galaxies
(iii) renewed expansion, recession accelerates
& new structure not likely to form.
De Sitter Model: $p=q=\lambda=0$

Not a rel. cosmo. since it is devoid of matter, but is of historical interest. Violates Mach's principle: (16) $\Rightarrow$

$$\frac{3}{R^2} - \Lambda = 0 \Rightarrow \frac{R}{\Lambda} = \left(\frac{4}{3}\Lambda\right)^{1/2}$$

Integrate & rescale to absorb constant: $R = \exp\left(\frac{1}{\sqrt{3}/\Lambda}\right)$

$$ds^2 = \exp\left[\frac{2t}{(3/\Lambda)^{1/2}}\right] [dx^2 + dy^2 + dz^2] - dt^2$$

Time-Scale Problem

Early estimate of $H_0 \approx 500$ km s$^{-1}$ Mpc$^{-1}$. $\Rightarrow T_0 = 1.8 \times 10^9$ yr.

Yet $T_{\odot} > 4 \times 10^9$ yr & $T_\odot > 10 \times 10^9$ yr.

Now $\dot{R} \leq 0 \forall t$ (i.e. $\dot{q} > 0$)

then: $t_0 - t_1 = \frac{R(t_0)}{R(t_0)} = t_0 - t_0 < t_0$

Today this may still be a problem:

if $H_0 > 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$, for at least some time. Could obviate if $R > 0$ for at least some time. Note that

$$(17) \times -3 + (16) \Rightarrow \frac{8\pi G (\rho + 3p)}{R^2} = 2A - 6B$$

Now LHS > 0 & if $\Lambda > 0 \forall t$. To allow $R > 0$ we may require $\Lambda > 0$ to fix the time-scale problem.
Standard Models: $\Lambda = 0$

After stellar nucleosynthesis understood & better Cepheid distances $\Rightarrow H_0 < 100$
the time-scale problem was thought to be unimportant, i.e. $T_o \sim 1.5 \times 10^{10}$ yr & $T_o < T_0$ is OK.

Let $\Lambda = 0$ in (37) so:

$$\ddot{R} = \frac{8}{3} \pi R^2 p(t) = \left[\rho(t) - \frac{3H_0^2}{8\pi} \right] \frac{8}{3} \pi R^2$$  (57)

Define the critical density:

$$\rho_c = \frac{3H_0^2}{8\pi}$$  (58)

If $T_0 = 10^{10}$ y then $\rho_c = 2 \times 10^{-29}$ g cm$^{-3}$

Now, inhomogeneous $\approx (0.02-0.03)\rho_c$: **MATTER**

From (36) we get:

$$\Omega_0 \approx \frac{4\pi R^2}{3H_0^2} = \frac{1}{2}\Omega = \frac{1}{2}\rho_c$$  (59)

Case: $k = 0$ $\Rightarrow$ Einstein-de Sitter:

$\rho = \rho_c$, $\rho_0 = \frac{1}{2}$ ... Has time scale & missing matter problems ... agrees w/inflation

Case: $k = +1$ $\Rightarrow$ Oscillating (or Big Crunch):

$\rho > \rho_c$, $\rho_0 > \frac{1}{2}$ ... Worse time-scale problem, worse missing matter problem, CLOSED, BOUND UNIVERSE

Case: $k = -1$ $\Rightarrow$ Indefinitely Expanding

$\rho < \rho_c$, $\rho_0 < \frac{1}{2}$ ... Can fix time-scale problem, don't need much or any dark matter, but $\rho_0$ is prob. too low:

OPEN, ALWAYS INFINITE UNIVERSE
Early Epochs: Pressure was important & hot enough that radiation dominated matter:

\[ T \propto \frac{1}{r}, \quad \rho \propto \frac{1}{r^3} \text{ but } \rho \propto \frac{1}{r^4} \quad \text{(60)} \]

\[ \therefore \rho = \frac{1}{3} \rho_r \quad \text{extreme rel. EoS} \quad \text{(61)} \]

Setting \( \Lambda = 0 \) in (16) & (17) & \( p(t) = \frac{1}{3} \rho(t) \Rightarrow \)

\[ \frac{\dot{R}^2}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{h^2}{R^2} = 0 \quad \text{(62)} \]

At early times 1st 2 terms dominate & \( \Lambda \) is irrelevant. Then, at small \( t \):

\[ R \propto t^{1/2} \quad \text{which is faster than the small } t \text{ behavior we had before (w/o p): } R \propto t^{2/3} \]

i.e. \( p \) exerts its own grav field strengthening collapse (in \( \star \)) and expansion (via time reversal).

Dirac Large Numbers

\[ \frac{R_{\text{universe}}}{R_{\text{electron}}} = \frac{c T_0}{e \gamma m_e c^2} \approx 10^{40} \quad \text{(63)} \]

\[ \frac{F_{\text{p, electric}}}{F_{\text{p, grav}}} \approx 2 \times 10^{39} \quad \text{(64)} \]

"Number of particles in Universe" = \( \frac{5}{m_p} (c T_0)^3 \approx 10^{79} \quad \text{(65)} \]

These coincidences \( \Rightarrow G \rho T^2 \approx 1 \quad \text{(66)} \)

Machian idea – just enough matter in universe to induce appropriate inertia in local body agrees w/ (66)!
Steady State Model

Perfect Cosmo Principle: Universe is Unchanging on a Large Scale ... must expand, for if static ... thermo $\Rightarrow H \neq 0$ & "heat death"
But $\rho = \text{const} \Rightarrow$ continuous creation of matter
- only at $3\pi H_0 \sim 10^{-46} \text{g cm}^{-3} \text{s}^{-1}$ so no problem.

(Outline: 1) Perfect Cosmo, 2) Weyl's Postulate
3) GR light propagation

- At least Robertson-Walker but stationarity $\Rightarrow R(t) \not\equiv$ curvature $\propto R^{-2}$ is observable; const
- $k = 0$: $H_0 = \frac{R'(0)}{R(0)}$ is obs. $\Rightarrow$ const $\Rightarrow \frac{a}{t} = \text{const}$
- $R(t) = \exp(\pm t)$ or:

$$ds^2 = \exp\left(\frac{2t}{\rho}\right) \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - dt^2$$

i.e. D'Inverno model — discarded before as empty but now OK, since full GR field eqns don't hold.

(3) Light prop $\Rightarrow \frac{dt}{\exp(\pm t/\rho)} = \pm dr$ (65)

For an incoming ray: $r = T_0 (e^{-t/\rho}e^{-t_0/\rho})$ &

$$d_L = r_0 e^{t_0/\rho} \text{ so rec. lum. dist.}$ Can then show:

$$d_L = \frac{z}{T_0} \text{ [Hubble Law]} \text{ is Exact}$$

$$f = -\frac{1}{3}\rho \text{ always expanding}$$

Field eqns: $C(x^a) \propto \text{OP - geodesic curve}$

$$G_{ab} = 8\pi T_{ab} + C_{ab}$$

& universal length: $C_{\alpha} = \frac{3}{T_0} \Rightarrow$ de Sitter (67) metric

$$D = \frac{3H_0^2}{8\pi} \Rightarrow \text{creation of matter via:}$$

M were Beyond $8\pi T_{ab} = -C_{ab}$ $\Rightarrow$ not patchy

Real Universe: $f > 0$ & Hubble Law inexact. Also $\rho \not= 0$ has evolved.
**Inflation**: From GUTs - strong electro-weak
Sym. broken @ $t \approx 10^{-34}$
Exponential expansion when energy density dominated by vacuum energy density of a scalar field

So $\rho \approx \rho_0, \Lambda = 0$ take (76) (77)

$$R^2 = \frac{8}{3} \pi \tau R^2 \rho_0 - k$$ (77)

At such early time $R^2$ term dominates, so:

$$\frac{\dot{R}^2}{R^2} = \frac{8}{3} \pi \tau \rho_0 = H^2$$ (72)

$$\Rightarrow R = R_0 \exp(HT)$$ (73) - deSitter like inflation.

Ending it is possible w/ dissipation in a phase transition from "false" to "true" vacuum.
Solves FLATNESS + HORIZON problems.

**Flatness**: now: $0.01 < \Omega_0 < 1.0$ by observation
In the past this required FINE TUNING
@ $t = 10^{-33}$ [1.2 - 1.1 x 10^{-57}]! But inflation forces extreme flatness - think of a balloon!

**Horizon**: now: homog. & isotropic. But only possible if true in early time. BUT standard model not causally connected, so A & B can't know of the other. Inflation means A is very closed could be connected at a very early time, then taken out of connection w/ exponential expansion.

Connects w/ ANTHROPIC PRINCIPLE & many universes idea.
Fig. 3.7: The complete history of the Universe.
Fig. 4.3: The development of primordial nucleosynthesis. The dashed line is the baryon density, and the solid lines are the mass fraction of $^4$He, and the number abundance (relative to H) for the other light elements.

earlier freeze out of the neutron-to-proton ratio: $T_F \propto g^1/6$, at a high value, and hence more $^4$He. Later we will use the dependence of $T_F$ up
4.5 Abundances: Observations

Fig. 4.4: The predicted primordial abundances of the light elements as a function of $\nu$. The error bar indicates the change in $Y_p$ for $\Delta\tau_{1/2} = \pm 0.2$ min.