VISCOSITY IN ACCRETION DISKS

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Abstract. Various proposals and prescriptions for the viscosity in accretion disks, which is needed to carry matter inward and angular momentum outward, are considered. The simplest picture, where the shear stress is taken as proportional to the local pressure, is clearly an oversimplification. A mechanism involving local magnetic instabilities generating turbulence may well provide a solution to this difficult problem.

1. Introduction

The nature of the viscosity within accretion disks remains one of the greatest uncertainties afflicting models for inflows onto compact objects. Given the presence of even a modicum of angular momentum in the accreting fluid, the strength of the viscosity will have a substantial influence upon the structure of the accretion disk. Unfortunately, under nearly all astrophysically relevant circumstances, all of the well understood microscopic transverse momentum transport mechanisms such as ionic, molecular, and radiative viscosity, are extremely small. Observations with direct relevance to the nature and strength of macroscopic viscosity mechanisms are very difficult to make and their interpretations are model dependent. In the case where a black hole is the compact accretor, such observations have not yet been possible.

Therefore, advances in understanding disk viscosity must combine theoretical analyses and numerical techniques. In this article we first review the simple α parameterization, where the shear stress is proportional to the total pressure in the disk; this was introduced by Shakura & Sunyaev (1973, hereafter SS73) in their classic paper 25 years ago, which effectively launched the subject. Thermal instabilities in such α disks led to the consideration of so-called β disks, where the shear stress is assumed to couple to only the gas pressure, and this modification is discussed next. Various ways that have been proposed to produce disk viscosity are then summarized. Over the past several years an explanation of viscosity in terms of linear magnetic instabilities (the Velikhov-Chandrasekhar-Balbus-Hawley instability) has been investigated intensely, and a summary of the numerical experiments which support this hypothesis is provided before conclusions are presented.

2. The α Parameterization

The "standard thin-disk" model involves little more than the equations of mass, angular momentum, and energy conservation. Within the approximate version introduced by SS73, as well as the fully general relativistic version put forward by Novikov & Thorne (1973), the only clearly open question is how to treat the torque acting between different radial zones of the disk. If we specialize to the steady Keplerian approximation of SS73, in which inertial terms and the pressure gradient are ignored (this is adequate for a discussion of the nature of the viscosity, though a major oversimplification with respect to the structure of the inner disk region), these conservation laws become:

$$\dot{M} = -2\pi r \Sigma v_r = \mathrm{const}, \tag{1}$$

$$\dot{M} r^2 \Omega = -2\pi r^2 W_{r\phi} + \dot{J}, \tag{2}$$

$$\frac{d}{dr}\Big[\dot{M}\Big(\frac{1}{2}v_\phi^2 - \frac{GM}{r}\Big) + 2\pi r^2 W_{r\phi}\Omega\Big] = 2\pi r Q^-. \tag{3}$$

In the above equations, $\dot{M}$ is the mass accretion rate, $\Sigma(r) \equiv \int \rho(r,z)dz$ is the column density, Ω is the angular velocity, $\dot{J}$ is the boundary flux of angular momentum, M is the mass of the central accretor, and $Q^-(r)$ is the rate at which heat is radiated from the disk. The integrated shear-stress which transfers the angular momentum outward through the disk as the mass spirals inward, is

$$W_{r\phi} \equiv \int t_{r\phi} dz = -\frac{3}{2}\Omega \int \eta dz, \tag{4}$$

in the Keplerian approximation, where η is the dynamical viscosity. The dominant local shear stress component (carrying azimuthal motions into radial ones) can be more generally written as $t_{r\phi} = \eta r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)$.

To take the models further demands the inclusion of an equation of state, usually taken to be the sum of gas and radiation pressure contributions,

$$P = P_\mathrm{g} + P_\mathrm{r} = \frac{k}{\mu m_\mathrm{H}}\rho T + \frac{a}{3}T^4; \tag{5}$$

however, possible contributions from magnetic fields, degenerate electrons and bulk motions (ram pressure) cannot be ignored under some circumstances. Then equations for the vertical structure can be derived (SS73), though we will not give them here. The heat generated in a column through the disk is

$$Q^{+} = \int t_{r\phi} r \frac{\partial}{\partial r}\Big(\frac{v_\phi}{r}\Big) dz = W_{r\phi} r \frac{\partial}{\partial r}\Big(\frac{v_\phi}{r}\Big);$$

we also define, $W \equiv \int P dz$.

The viscosity prescription is "derived" by noting that in the Navier-Stokes equations the turbulent contribution to the force, $F_{\rm vis} \approx \eta \frac{v_{\rm turb}}{L_{\rm turb}}$, in terms of the dynamical viscosity and the typical eddy velocity and size. In that very high Reynolds numbers are found for any plausible astrophysical fluids, the turbulence should be strong. Therefore, $F_{\rm vis} \approx F_{\rm inertial} \simeq \rho v_{\rm turb}^2$, which implies $\eta_{\rm turb} \approx \rho v_{\rm turb} L_{\rm turb}$. Shakura & Sunyaev argue that if the turbulence became supersonic, shocks would ensue between colliding elements, thereby directly converting bulk kinetic energy to heat; this plausible assumption yields the constraint, $v_{\rm turb} < c_s$, the sound speed. Further, assuming isotropic turbulence, the maximal eddy size is limited by the half-thickness of the disk, z_0. Therefore,

$$-t_{r\phi,{\rm turb}} \simeq \eta_{\rm turb}\Omega < \rho c_s z_0 \Omega \approx \rho c_s^2. \tag{6}$$

A similar limit can be placed on the shear stress carried by local magnetic fields (SS73):

$$-t_{r\phi,{\rm mag}} < P_{\rm mag} = B^2/8\pi < P \approx \rho c_s^2. \tag{7}$$

Together, equations (6) and (7) lead to the α-disk prescription,

$$|t_{r\phi}| = \alpha P, \quad \text{or} \quad W_{r\phi} = \alpha W, \tag{8}$$

using equation (4). Within the Keplerian approximation this implies,

$$\eta = \frac{2}{3}\alpha P \Big(\frac{GM}{r^3}\Big)^{-1/2}, \tag{9}$$

with the α parameter (clearly $\alpha \leq 1$) usually assumed to be a constant throughout the disk, though this constancy is obviously unlikely to hold in reality. Various physical parameters of a disk are then functions of α, though fortunately, most of the dependences are rather weak. For example, in the so-called "middle" region of a thin disk, where $P_{\rm g} > P_{\rm r}$, and the dominant opacity is from electron scattering: $(z_0/r) \propto \alpha^{-1/10}$, the disk's central temperature $\propto \alpha^{-1/5}$, while the disk's central density $\propto \alpha^{-7/10}$ (SS73). Despite the obvious crudeness of this α-disk model, its simplicity has proven to be wildly popular, and certainly over 90% of the more than

6000 papers written on accretion disks over the past 25 years have employed this approximation.

3. The β Parameterization

Shortly after the α-models were proposed, a problem with them was noticed: the "inner region", where $P_{\rm r} > P_{\rm g}$ is unstable (Lightman & Eardley 1974, hereafter LE; Shakura & Sunyaev 1976, hereafter SS76). These instabilities result when a time-dependent analysis is performed. We shall not present much of the mathematics, but will merely summarize the results. It is also worth keeping in mind that it is in this inner region where the Keplerian approximation, in particular its assumption that pressure gradient forces can be ignored, is most likely to fail, thereby providing the impetus for studies of more general accretion flows (e.g., Paczyński & Wiita 1980; Chakrabarti 1990).

When $P_{\rm r} \gg P_{\rm g}$, P is independent of ρ, but the equation of vertical pressure balance against the out-of-plane gravitational forces of the black hole yields $P \approx 0.5 z_0 \Omega^2 \Sigma$. Combining this with equation (4) yields the peculiar result (LE) that $W_{r,\phi}(\Sigma, r) \propto \Sigma^{-1}$. Because the integrated stress is a decreasing function of the column density, the non-linear diffusion equation for Σ, which is (LE),

$$\frac{\partial \Sigma}{\partial t} = \frac{\partial}{\partial r}\Big[\frac{d(\Omega r^2)}{dr}\Big]^{-1} \frac{\partial}{\partial r}[r^2 W_{r,\phi}(\Sigma, r)], \tag{10}$$

has a *negative* effective diffusion coefficient. Physically, this means that an initially stationary inner region will start to break up into rings (with $\Delta r \geq z_0$); one will have high-Σ and low-$W_{r,\phi}$ while its inner- and outer-neighbors will have low-Σ and high-$W_{r,\phi}$. Thus, the matter gets pushed into rings of low viscous stress while the alternate rings become so starved of matter that they become optically thin, and therefore unstable. Such a dynamical instability would naturally lead to rapid fluctuations in the emission from the inner portion of the disk, perhaps explaining some of the fluctuations observed in many X-ray binaries and active galactic nuclei.

It is easily seen that this instability could be avoided if the viscosity is a function of Σ (or r) rather than constant; in particular, if α falls at least as fast as Σ^{-1} then $\Sigma(r)$ rises rapidly near the black hole and the instability does not occur. In particular, Cunningham's (1973) suggested β-disk prescription,

$$t_{r\phi} = \beta P_{\rm g}, \tag{11}$$

is stable to this LE clumping. Such "β-disks" are reasonably similar to α-models, except that they have much higher values of Σ in their inner regions and thereby remain optically thick despite staying geometrically thin.

A more complete linear analysis was performed by SS76, who considered axisymmetric perturbations of wavelength Λ in a thin disk, such that $z_0 \ll \Lambda \ll r$; these perturbations are assumed to grow slowly compared to the dynamical timescale, which is locally given by Ω^{-1}. They found that the LE dynamical instability actually corresponded to the more slowly growing branch of a general dispersion relation. If one defines $P_\mathrm{r} = \chi P$, the result of the SS76 analysis is that for $\chi < 0.6$ (i.e., gas pressure dominance), disk accretion is stable. For $\chi > 0.6$ there exists an unstable mode for $\Lambda/z_0 > f(\chi)$; for $\chi = 1, f(\chi) = 2$ (its minimum value), and this mode corresponds to rings moving across the disk with $2z_0 < \Lambda < 4z_0$. The growth rate, $\omega = 0$ at $\Lambda = 2z_0$ but is up to $\omega \simeq \alpha\Omega/10$ at $4z_0$. For longer wavelengths, the solution bifurcates into slower and faster growing branches. The lower (LE) branch corresponds to the situation where viscosity perturbation amplitudes $(\delta\eta/\eta)$ are much less than $\delta\Sigma/\Sigma$ or $\delta z_0/z_0$. Under these circumstances, the rate of heat loss can balance the rate of production, i.e., $Q^- = Q^+$, and the growth rate decreases as the wavelength of the perturbation increases,

$$\omega \simeq \alpha\Omega \frac{2(5-3\chi)}{3(5\chi-3)}\left(\frac{z_0}{\Lambda}\right)^2. \tag{12}$$

The faster growing thermal instability has $Q^+ \neq Q^-$, and in this situation $\delta\Sigma/\Sigma \ll (\delta z_0/z_0, \delta\eta/\eta)$; here the growth rate is given by

$$\omega = \alpha\Omega \frac{6(5\chi-3)}{A(\chi)}, \tag{13}$$

where $A(\chi) = P/(e\rho)$, with e the specific internal energy; for a $\gamma = 5/3$ gas, $A = 3(1+\chi)/2$. As $\chi \longrightarrow 1$ this perturbation growth rate is $\omega \simeq 0.21\alpha\Omega$ for $\Lambda > 12z_0$, and is therefore quite fast as long as α is not negligibly small.

If the viscosity only couples to the gas pressure, which is essentially equivalent to saying it is dominated by actual mass motions, and not magnetic fields, then the β prescription of $t_{r\phi} = \beta P_g$ can be analyzed in the same way. Such β disks are completely stable to these linear perturbations. Various authors have considered aspects of these β disks, among them Meyer & Meyer-Hofmeister (1982), Wandel & Petrosian (1988), and Mangalam & Wiita (1993). Key differences are that the β disks have somewhat larger central and surface temperatures and somewhat smaller thicknesses than do α disks for the same M and $\dot{M}$.

4. Possible Physical Mechanisms for Viscosity

In this section we briefly mention most of the mechanisms that have been suggested to provide viscosity in disks, but by no means do we attempt to provide a thorough review. Despite the crudity of the α model, it does provide a useful way of talking about the strength of viscosity, and we will summarize the viscosity mechanisms in terms of the values of the α parameter that they apparently produce.

4.1. HYDRODYNAMICAL MECHANISMS

One possibility is that the mere differential rotation of the disk will generate turbulence through shear instabilities. While two-dimensional laboratory experiments indicate that sheared cylindrical flows are indeed unstable, this possibility has remained very controversial when considered in the realm of accretion disks, since locally Keplerian flows are linearly stable (Stewart 1975). Non-linear HD instabilities may well be present (e.g., Dubrulle & Knobloch 1992), but only rather low (probably $< 10^{-2}$) values of α are anticipated from this possible mechanism.

A hydrodynamical modification on the basic SS73 idea of largest eddy turbulence asks if their limit, $L_{\rm turb} \leq z_0$, is absolute. It is distinctly possible that the turbulent cells are elongated by the shear and accretion flows so that the turbulence is not isotropic. If the disk turbulence is sufficiently anisotropic, $L_{\rm turb} \gg z_0$ is not inconceivable. This could produce, in principle, a very high value of $\alpha > 1$.

One way of guaranteeing the presence of turbulence in a disk is to have a need for convection to transport energy vertically through the disk; in analogy with stellar convection turbulence would then be clearly present, with a maximum eddy size corresponding the the thickness of the convection layer. Early estimates claimed that convection could be important (Bisnovatyi-Kogan & Blinnikov 1977), but later calculations (Meyer & Meyer-Hofmeister 1982) indicated that most disks are convectively stable. Even if convection is present and important for energy transport, it would limit the turbulence to subsonic speeds and probably smaller scales than z_0 in most parts of the disk. This implies low values of α if convection is the dominant source of viscosity.

Since galactic X-ray sources are in binary systems, it is well worth considering the influence of the binary companion on the accretion disk. Many authors have examined this possibility, and it is quite clear that a binary companion can drive nearly stationary spiral shocks through the disk (e.g., Sawada, Matsuda & Hachisu 1987; Różyczka & Spruit 1993; Chakrabarti & Wiita 1993). In some cases these shocks provide very strong transport of angular momentum and allow adiabatic accretion with predicted values for $\alpha(r)$; these tend to be rather high, with $\bar{\alpha} \sim 0.1$.

Other interesting HD mechanisms that might be relevant to the generation of effective viscosity include inertial waves (e.g., Vishniac, Jin & Diamond 1990) and tidal forces driving small scale turbulence (e.g., Ryu & Goodman 1994), both of which might produce modest values of α.

4.2. MAGNETOHYDRODYNAMICAL MECHANISMS

There are several ways in which magnetic fields might dominate the viscosity and angular momentum transport. In this subsection we mention a few relatively speculative possibilities based on large scale fields; in the next section we discuss a mechanism that is now believed to be of key importance.

The possibility that a large-scale dynamo can be established in an accretion disk is worthy of consideration, and, while generally considered unlikely, it appears that this is possible under some circumstances, (e.g., Mangalam & Subramanian 1994). If so, large B fields could give rise to strong shear and reconnection. While such putative dynamo fields are likely to be important for producing heating and particle acceleration, there is no obvious way to derive the viscosity; nonetheless, the presence of an effective dynamo should yield a rather large value of α.

Many authors have suggested that a corona exists above and below accretion disks (e.g., Galeev, Rosner & Vaiana 1979; de Vries & Kuijpers 1992), though the lack of convection zones may mean that flux tubes cannot be anchored and may well escape without forming stellar-like coronæ (Chakrabarti & D'Silva 1994). If magnetically heated coronae do exist, then substantial dissipation of energy and angular momentum is possible through reconnection and flares in the corona. This is likely to produce moderate effective viscosities, with $\alpha \sim 0.1$.

Magnetic loops could stretch between annular rings at substantially different radii in an accretion disk, and the difference in velocity could shear and amplify the field. Locally driven winds as well as loops tying together different radii can very effectively transport angular momentum. In principle, this type of coronal torque could dominate, with (Burm & Kuperus 1988)

$$\frac{\text{coronal torque}}{\text{body torque}} = \Big(\frac{B_z B_\phi/8\pi}{\alpha P}\Big)\Big(\frac{r}{z_0}\Big) \approx \frac{r}{\alpha z_0} \gg 1. \tag{14}$$

This could produce the equivalent of a medium to very large α; however, as in the case of global dynamos and coronæ, there is no guarantee that this type of MHD process plays a significant role in actual accretion disks.

5. Magnetic Instabilities Yield an Effective Viscosity

We finally turn to a form of magnetic viscosity that almost certainly is present in all accretion disks.

5.1. THEORETICAL ANALYSIS

A linear magnetic instability for magnetized Couette flow has been known for a long time (Velikhov 1959; Chandrasekhar 1961), but its key importance for accretion disks was stressed by Balbus & Hawley (1991, hereafter BH) and it is therefore usually called the Balbus-Hawley (B-H) instability. This instability only requires the presence of a very weak seed field, which could be advected in with the accreted material, anchored in the central mass (especially if it is a neutron star), or generated by a dynamo in the disk itself (e.g., Brandenburg et al. 1995).

The basic idea is simple (BH): consider a weak vertical field, B_z, in a disk undergoing differential rotation. Any small perturbation outward (in the r direction) grows rapidly (on the dynamical, Ω^{-1}, timescale). To see this, consider an outwardly displaced fluid element; the magnetic field simultaneously tries to: (i) enforce rigid rotation (by resisting shearing); (ii) return the element back to its original position (by resisting stretching). While the latter aspect is patently stabilizing, the former produces the instability, by forcing the element to rotate too fast for its displaced location. Since its velocity is too large, the inertial ("centrifugal") force pushes the element yet further outward. For short enough wavelengths the restorative force wins, but for long enough wavelengths the instability grows.

The B-H instability is clearly effective in transporting angular momentum outward from the faster inner region to the slower outer region. The linear analysis produces the following criterion for it to operate (in the limits that the disk is rotating supersonically or if the Brunt-Väisälä frequency is negligible; for the full expression, see BH):

$$k_z^2 v_{zA}^2 < \frac{d\Omega^2}{d \ln r}, \tag{15}$$

where k_z is the wavenumber in the vertical direction and v_{zA} is the vertical component of the Alfven speed, $v_{zA} = B_z/(4\pi\rho)^{1/2}$. The B-H instability also effectively transports magnetic flux inward, as can be shown using interchange stability arguments (Christodoulou, Contopoulos & Kazanas 1996). Although this instability in its simplest form depends upon the presence of a vertical field component, and the Keplerian and accretion flows in the disk tend to amplify the azimuthal and, to a lesser extent, the radial, components without affecting B_z, it is almost inconceivable that any realistic situation will not have at least *some* non-zero B_z. The most rapidly growing wave numbers have growth rates around 0.75Ω, implying that even minute fields will be amplified by factors of 10^6 in under 3 rotation periods. Another key result is that these growth rates are independent of the strength of the magnetic field, at least for the linear analytic theory (BH).

Additional work showed that a purely toroidal field also leads to instability (Balbus & Hawley 1992; Foglizzo & Tagger 1995); in this case the dominant restoring force involves pressure gradients. In both cases, the mechanism leading to the instability arises from the fact that when two fluid elements in the same Keplerian orbit are pulled together, they are torn apart in the radial direction due to the change in the centripetal acceleration. As noted by Brandenburg et al. (1996), this problem is equivalent to the difficulties encountered by spacecraft attempting docking maneuvers. Once the elements are at different radii, but coupled by either originally vertical or azimuthal fields, angular momentum exchange can occur and energy can be converted from the shear motion into heat through a turbulent cascade. In the limit where the fluid is locally incompressible, Goodman & Xu (1994) discovered an exact solution of the axisymmetric magnetic shearing instability that grows exponentially even in the non-linear stage.

5.2. NUMERICAL RESULTS

Clearly such analytical results demand numerical investigation to see if they survive non-axisymmetric perturbations and to see in what fashion(s) they saturate. A compressible two-dimensional MHD computation was immediately presented by Hawley & Balbus (1991), who showed that for weak fields, the numerical results were consistent with the linear perturbation theory. However, since the generation of magnetic fields is an inherently three-dimensional phenomenon, such restricted computations could not be convincing. Quite a few additional investigations have been performed and the general applicability of the instability under a wide range of conditions has now been demonstrated.

A very important step involved going to three-dimensional MHD simulations. One approach is that of Hawley, Gammie & Balbus (1995), who considered a "local" disk (shearing box) model: one where tidal and Coriolis forces are incorporated but background gradients in pressure and density are ignored, along with the vertical component of gravity; periodic boundary conditions were employed in the radial direction. They found that, regardless of whether the computations start with only vertical or only azimuthal fields (or with a combination of both), turbulence is initiated and sustained by the magnetic instability. The turbulence saturates and is anisotropic in the sense that angular momentum is transported outwards and the turbulent energy and angular momentum flux are concentrated on the largest scales. Stone et al. (1996) added vertical gravity and density stratification to this code and found quite similar results.

Three-dimensional simulations were also carried out by Matsumoto & Tajima (1995), who also used the shearing-box model but employed a different algorithm for the evolution of the magnetic field. They mainly studied initially purely toroidal fields and also saw the development of strong turbulence. Their results also indicate that the magnetic fluctuations saturate at late stages, and they explain this as arising from the enhanced resistivity produced by tangled field lines. Their relatively coarse simulations yield equivalent $\alpha_B \equiv -\langle B_x B_y \rangle / (4\pi P)$ values typically around 0.02 for initially toroidal fields, but they claim α_B can exceed 0.1 for initially vertical seed fields.

Yet another three-dimensional local simulation was produced by Brandenburg et al. (1995). Their model also included the effects of compressibility and stratification, but employed a significantly different algorithm, which is of great importance in checking such complex computations. Brandenburg et al. also dispensed with the periodic vertical boundary conditions used by Hawley et al. (1995) Stone et al. (1996) and Matsumoto & Tajima (1995), as these implied that the total magnetic flux through the box was always conserved at zero. Instead they assume that the magnetic field at the upper and lower faces of the box is vertical, but that the field lines can move freely across the surface, which allows for generation of a net magnetic field. They find that supersonic flows are first driven by the B-H instability; these flows regenerate a turbulent magnetic field which then reinforces the turbulence. Therefore, this system can be characterized as a dynamo, but one where the magnetic energy exceeds the turbulent kinetic energy, typically by factors of 3–10. Their original simulations yielded effective α values of between 0.001 and 0.005; these computations have the Maxwell (magnetic) stress corresponding to $\sim$5 times as much viscosity as the Reynolds (turbulent) stress.

More recent work by Brandenburg et al. (1996), improved their model by removing scale invariance and allowing finite accretion rates. This nonlocal model is more realistic and several different resolutions were used. They estimated the α parameter in three distinct ways: (i) via the mass accretion rate; (ii) using the heating rate; (iii) through horizontal components of the Maxwell and Reynolds stress tensors. All of these approaches were quite consistent, though now they found $\alpha_{\rm Maxw}/\alpha_{\rm Reyn} \approx 3$. When the numerical resolution is doubled, the effective values of α go up by factors of ~ 1.5 within the range they could investigate, and the highest resolution runs yielded $\alpha \approx 0.007$. The latest results from this program at slightly higher resolution (Brandenburg 1998) still yield $\alpha < 0.01$, though it remains to be seen if this value really saturates or if continued improvements in computational capabilities yield additional increases in the effective viscosity.

6. Conclusions

The only fairly direct observational evidence for the strength of disk viscosity comes from dwarf novae: binary systems where a white dwarf accretes through a disk. Substantial luminosity is released in a hot spot, where the accretion stream from the companion impinges on the outer portion of the disk, which complicates the analysis, but the spectrum can sometimes be confidently divided between the white dwarf, the companion, the disk and the hot-spot. These systems undergo strong, quasi-periodic outbursts. This large scale variability has long been convincingly understood in terms of thermal limit cycles behavior in the accretion disk when hydrogen recombination occurs (e.g., Meyer & Meyer-Hofmeister 1981; Cannizzo, Ghosh & Wheeler 1982; Mineshige & Osaki 1983). When these systems undergo eclipses, careful analyses of the light curves allow the temperature profile, $T(r)$ of the disk to be estimated. During quiescent phases, a flat brightness–temperature profile curve is found, while during outbursts, $T_{\rm eff} \propto r^{-3/4}$ is more usual. Both these temperature variations and the relative lengths of the bursting and quiescent stages can be fit to the theory if $\alpha \sim 0.1$ in the bright phase and $\alpha \sim 0.02$ in the quiescent phase.

As mentioned in the introduction, no such observational evidence is yet available when black holes are the accreting masses, and the physical conditions in the disks around even stellar remnant black holes will be different from those around white dwarfs, not to mention the large differences for the parameters expected in disks around supermassive black holes. Nonetheless, such values of α are not unreasonable, and we have seen that a wide array of physical mechanisms are capable of producing effective viscosities of that order. Furthermore, it now appears that the local magnetic instabilities must yield some non-trivial amount of angular momentum transport. As has been seen from the discussion in §5.2, while there is general agreement on the viability and importance of the magnetic instabilities in accretion disks, there is some disagreement between groups as to the effective strength of the viscosity. So, although final estimates are not yet available, values of $\alpha \sim 0.01$ are certainly likely from this mechanism, and somewhat higher values are possible.

Therefore, even though there is still a substantial amount of ignorance concerning the complete nature of and strength of viscosity in accretion disks, we can now argue that taking $0.005 < \alpha < 0.3$ is a safe bet and that further constraining this parameter towards the geometrical middle of that range is not unreasonable. This is a major step forward, and it seems likely that a very good prescription for viscosity will be available long before the theory of accretion disks is 50 years old.

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